ON CERTAIN BASIC HYPERGEOMETRIC SERIES TRANSFORMATIONS AND SUMMATIONS

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By
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Certificate

This is to certify that the thesis entitled **On Certain Basic Hypergeometric Series Transformations and Summations** submitted by Syed Nadeem Hasan Rizvi for the award of the Degree of Doctor of Philosophy in Mathematics by Babu Banarasi Das University, Lucknow is a record of authentic work carried out by him under my supervision. To the best of my knowledge, the matter embodied in this thesis is the original work of the candidate and has not been submitted elsewhere for the award of any other degree or diploma.

June 30, 2015

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Declaration

I, Syed Nadeem Hasan Rizvi, hereby declare that the work presented in this thesis, entitled **"ON CERTAIN BASIC HYPERGEOMETRIC SERIES TRANSFORMATIONS AND SUMMATIONS"** in fulfillment of the requirements for the award of Degree of Doctor of Philosophy of Babu Banarasi Das University, Lucknow, is an authentic record of my own research work carried out under the supervision of Dr. S. Ahmad Ali. I also declare that the work embodied in the present thesis is my original work and has not been submitted by me for any other Degree or Diploma of any university or institution.

Date: 30th June 2015 **(Syed Nadeem Hasan Rizvi)**

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Syed Nadeem Hasan Rizvi

Preface

The present thesis entitled "On Certain Basic Hypergeometric Series Transformations and Summations" is the outcome of the researches carried out by me under able guidance and supervision of Dr. S. Ahmad Ali, Head, Department of Mathematics, Babu Banarasi Das University, Lucknow.

The thesis is divided into six chapters. In chapter 1, we give a brief background and some of the developments in the theory of generalized basic hypergeometric series, which form the subject matter of the present thesis. Chapter 2, contains some new transformations of basic and poly-basic hypergeometric series that have been established by using Bailey lemma. In the chapter 3, we have used the concept of Bailey chain to discover a number of new Bailey pairs. These Bailey pairs have been used to establish a number of basic hypergeometric series identities. In chapter 4, we have established certain identities connecting the two components of Bailey pair. We then have used some known Bailey pairs to drive a number of new summation and transformation identities of basic hypergeometric series. In chapter 5, we have given the bilateral extension of some known transformations of unilateral series using Cauchy's method of bilateralization. The last chapter 6, of the thesis contains some miscellaneous transformations and summation of basic hypergeometric series.

A shortened version of the thesis is contained in the following research paper

1) Certain Transformations and Summations of Basic Hypergeometric Series, J. Math. Comp. Sci., Vol. 5 (2015), 25-33. (with S. Ahmad Ali)

2) On Certain Transformations of Bilateral Basic Hypergeometric Series, accepted in J. Adv. Math. Appl. (with S. Ahmad Ali)

3) Certain Transformations of Basic and Poly-Basic Hypergeometric Series, communicated in Italian J. Pure Appl. Math. (with S. Ahmad Ali)

4) Certain New WP-Bailey Pairs and Basic Hypergeometric Identities, communicated in Scientia Series A: Math. Sci. (with S. Ahmad Ali)

5) Some Miscellaneous Transformations of Unilateral and Bilateral Basic Hypergeometric Series, communicated in South East Asian J. Mathematics and Mathematical Sci. (with S. Ahmad Ali).

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

The importance of the study of generalized hypergeometric functions lies in the fact that they combine, as particular case, most of the special functions of analysis. Thus, a formula derived for the generalized hypergeometric function, or more general functions defined through it, becomes a master formula and provides a handy tool for obtaining results for other simpler functions. During the last six decades the generalized basic hypergeometric series have proved to be particularly important, since they have valuable applications in theory of numbers, theory of partition, combinatory analysis and many branches of social, physics, natural and engineering sciences. The colossal mass of literature on basic hypergeometric series has become so significant and important that their study has acquired and autonomic and respective status of its own rather than merely being treated as a generalization of the ordinary hypergeometric series despite of the fact that form some extent the basic hypergeometric functions are generalization of ordinary hypergeometric functions.

For a complete account of the theory of basic hypergeometric series one may refer to the works of Bailey [24], Rainville [67], Slater [87], Exton [39, 40, 41], Srivastava et al [89, 90], Agarwal [2, 3], Gasper and Rehman [44], Fine [42], Andrews [12] and Andrews, Askey and Roy [20].

1.2 Notations and Definitions

In 1813, Gauss [47] introduced the ordinary hypergeometric series by means of the following infinite series

$$1+\frac{ab}{1\ c}z+\frac{a(a+1)b(b+1)}{1.2\ c(c+1)}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3\ c(c+1)(c+2)}z^3+........ \quad (1.2.1)$$

which can be written as

$${}_2F_1\left[\begin{array}{ccc} a, & b; & z \\ c & & \end{array}\right]=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n, \quad (1.2.2)$$

where 'c' is neither zero nor a negative integer.

The generalization of (1.2.2) with r numerator and s denominator parameters is given by

$${}_rF_s\left[\begin{array}{cccccc} a_1, & a_2, & ..... & a_r & ; & z \\ b_1, & b_2, & .... & b_s & & \end{array}\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n....(a_r)_n}{(b_1)_n(b_2)_n..(b_s)_n n!}z^n. \quad (1.2.3)$$

The series (1.2.3) converges

(i) for all finite z if $r \leq s$,

(ii) for $|z| < 1$, if $r = s+1$ and also when $z = 1$, provided $Re[\Sigma(b) - \Sigma(a)] > 0$; and when $z = -1$, provided $Re[\Sigma(b) - \Sigma(a) + 1] > 0$,

(iii) diverges for all $z \neq 0$, if $r > s + 1$.

Here, $(a)_n$ denotes the Pochhamer's symbol or shifted factorial, defined as

$$(a)_n = \begin{cases} a(a+1)(a+2).........(a+n-1); n \in N \\ 1; n = 0. \end{cases}$$

For $r = s + 1$, the series (1.2.3) is called *balanced or Saalschützian*, when $\Sigma(b)_s - \Sigma(a)_{s+1} = 1$ and *well-poised* when

$$1 + a_1 = a_2 + b_1 = ... = (a)_{s+1} + (b)_s.$$

Series (1.2.3) is said to be *nearly-poised of first kind* and *nearly-poised of second kind* respectively if

$$1 + a_1 \neq a_2 + b_1 = ... = (a)_{s+1} + (b)_s$$

and

$$1 + a_1 = a_2 + b_1 = ... = (a)_s + (b)_{s-1} \neq (a)_{s+1} + (b)_s.$$

As already remarked that ordinary hypergeometric series combine, as a special case, most of the special functions and polynomials, yet it does not contain, as a particular case, the elliptic and associated functions. This limitation is overcome through the introduction of generalised basic hypergeometric series. Heine [48] in 1898 defined the basic analogue of (1.2.1) as

$$1+\frac{(1-q^a)(1-q^b)}{(1-q).(1-q^c)}z+\frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2).(1-q^c)(1-q^{c+1})}z^2+........ \quad (1.2.4)$$

where q is real or complex, $q \neq 1$ and $c \neq 0, -1, -2.....$. The series in (1.2.4) is known as basic (or $q-$) hypergeometric series or Heine series. By taking $q \to 1$ in (1.2.4) we recover the series (1.2.1) by noting that

$$\lim_{q\to 1}\frac{1-q^a}{1-q} = a.$$

The series (1.2.4), for convenience of notation, is usually written as

$${}_2\varphi_1\left[\begin{array}{cccc} a, & b; & q, & z \\ & c & & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}z^n = \sum_{n=0}^{\infty}\frac{(a,b;q)_n}{(q,c;q)_n}z^n.$$

The generalization of Heine's series with r numerator parameters and s denominator parameters is given by [44]

$${}_r\varphi_s\left[\begin{array}{ccccc} a_1, & a_2, & ...a_r; & q, & z \\ b_1, & b_2, & ...b_s & & \end{array}\right]$$
$$= \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n....(a_r;q)_n}{(q;q)_n(b_1;q)_n(b_2;q)_n....(b_s;q)_n}z^n[(-1)^n q^{\frac{n(n-1)}{2}}]^{1+s-r} \quad (1.2.5)$$

where

$$(a;q)_n = \begin{cases} (1-a)(1-aq)(1-aq^2).....(1-aq^{n-1}); n \in N \\ 1; n = 0. \end{cases}$$

or equivalently,

$$(a;q)_n = \prod_{j=0}^{\infty} \frac{(1-aq^j)}{(1-aq^{n+j})} = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

and the series (1.2.5) converges

(i) for all z if $r \leq s$,

(ii) if $r > s+1$, the series converges only when $z = 0$,

(iii) for $|z| < 1$ if $r = s+1$,

(iv) converges absolutely if $|q| > 1$ and $|z| < \frac{|qb_1b_2...b_s|}{|a_1a_2....a_r|}$.

For $r = s+1$, the series ${}_r\varphi_s(.)$ is called *balanced* or *Saalschützian* if

$$qa_1a_2...a_{s+1} = b_1b_2...b_s$$

and if

$$qa_1 = b_1a_2 = ... = b_sa_{s+1} \tag{1.2.6}$$

series (1.2.5) is called *well-poised*. The series ${}_{s+1}\varphi_s$ is called *nearly-poised* if all but one of the pairs of parameter in (1.2.6) have the same product. The series (1.2.5) is known as *nearly-poised of first kind* if

$$qa_1 \neq a_2b_1 = ... = a_{s+1}b_s.$$

Same series is said to be *nearly-poised of second kind* if

$$qa_1 = a_2b_1 = ... = a_sb_{s-1} \neq a_{s+1}b_s.$$

The series (1.2.5) is called *very-well-poised* if it is *well-poised* and $a_2 = q\sqrt{a_1}$, $a_3 = -q\sqrt{a_1}$, $b_1 = \sqrt{a_1}$ and $b_2 = -\sqrt{a_1}$ respectively. Moreover, the series (1.2.5) terminates if one of its numerator parameter is of the form of q^{-m} as

$$ {}_2\varphi_1\left[\begin{matrix} q^{-m}, & b & ;q, & z \\ & c & & \end{matrix}\right] = \sum_{n=0}^{m}\frac{(q^{-m};q)_n(b;q)_n z^n}{(q;q)_n(c;q)_n} = \sum_{n=0}^{m}\frac{(q^{-m},b;q)_n}{(q,c;q)_n}z^n. $$

It is assumed a denominator can never be of the form q^{-m} as

$$ (q^{-m};q) = 0; \qquad\qquad n = m+1, m+2, ... $$

A Polybasic hypergeometric series is defined as

$$ \Phi\left[\begin{matrix} a_1 & ... & a_r & : & (c)_{1,1},..., & (c)_{1,r_1} & : ... : (c)_{m,1},..., & (c)_{m,r_m} & ; \\ b_1 & ... & b_s & : & (d)_{1,1},..., & (d)_{1,s_1} & : ... : (d)_{m,1},..., & (c)_{m,s_m} & \end{matrix}\right. $$

$$ \left. \begin{matrix} q, q_1, ..., q_m; & z \\ & \end{matrix}\right] $$

$$ = \sum_{n=0}^{\infty}\frac{(a_1, a_2....a_r;q)_n}{(q, b_1, b_2....b_s)_n}z^n[(-1)^n q^{\frac{n(n-1)}{2}}]^{1+s-r} $$

$$ \times\prod_{j=1}^{m}\frac{(c_{j,1},....,c_{j,r_j};q_j)_n}{(d_{j,1},....,d_{j,s_j};q_j)_n}[(-1)^n q_j^{\frac{n(n-1)}{2}}]^{s_j-r_j} $$

which converges for max $(|q|, |q_1|, ...|q_m|) < 1$.

The generalized basic bilateral hypergeometric series is defined as

$$
{}_r\psi_r\left[\begin{array}{cccccc} a_1, & a_2, & ...., & a_r; & q, & z \\ & & & & & \\ b_1, & b_2, & ...., & b_r & & \end{array}\right]
$$

$$
= \sum_{n=-\infty}^{\infty} \frac{(a_1;q)_n(a_2;q)_n...(a_r;q)_n}{(b_1;q)_n(b_2;q)_n...(b_r;q)_n} z^n, \quad (1.2.7)
$$

which converges $|\frac{b_1 b_2 ... b_r}{a_1 a_2 ... a_r}| < |z| < 1$. The series ${}_r\psi_r$ is called *well-poised* if

$$
a_1 b_1 = a_2 b_2 = ... = a_r b_r.
$$

It is called *very-well-poised* if it is *well-poised* and $a_1 = -a_2 = qb_1 = -qb_2$.

A *very-well-poised* ${}_r\psi_r$ series is known as *very-well-poised-balanced* if

$$
(a_3 a_4 .... a_r) qz = (\pm a_1 q^{-1/2})^{r-2},
$$

with either $+$ or $-$ sign and a *well-poised* ${}_r\psi_r$ series is called *well-poised-balanced* if

$$
(a_1 a_2 .... a_r) z = (\pm (a_1 b_1)^{1/2})^r,
$$

with either $+$ or $-$ sign.

In our work, we shall use the following q-shifted factorial identities frequently

$$
(a;q)_{-n} = \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n q^{n(n-1)/2}}{(q/a;q)_n}
$$

$$
(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)
$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

$$(aq^{-n};q)_n = (q/a;q)_n(-a/q)^n q^{-n(n-1)/2}$$

$$\frac{(aq^{-n};q)_n}{(bq^{-n};q)_n} = \frac{(q/a;q)_n}{(q/b;q)_n}\left(\frac{a}{b}\right)^n$$

$$\frac{(a;q)_{n-r}}{(b;q)_{n-r}} = \frac{(a;q)_n}{(b;q)_n}\frac{(q^{1-n}/b;q)_r}{(q^{1-n}/a;q)_r}\left(\frac{b}{a}\right)^r$$

$$(a;q)_{n+r} = (a;q)_n(aq^n;q)_r$$

$$(a;q)_{kn} = (a;q^k)_n(aq;q^k)_n.....(aq^{k-1};q^k)_n$$

$$(a^2;q^2)_n = (a;q)_n(-a;q)_n,$$

$$(a^3;q^3)_n = (a;q)_n(a\omega;q)_n(a\omega^2;q)_n, \qquad \omega = e^{2\pi i/3}$$

in general,

$$(a^k;q^k)_n = (a;q)_n(a\omega_k;q)_n(a\omega_k^2;q)_n......(a\omega_k^{k-1};q)_n, \qquad \omega = e^{2\pi i/3}$$

$$\frac{(qa^{1/2};q)_n(-qa^{1/2};q)_n}{(a^{1/2};q)_n(-a^{1/2};q)_n} = \frac{(aq^2;q^2)_n}{(a;q^2)_n} = \frac{1-aq^{2n}}{1-a}.$$

1.3 Transformations and Summations of Basic Hypergeometric Series

In the literature there exists a large number of identities connecting two or more basic hypergeometric series and also the summation identities. One of the most celebrated such identity is

$$ {}_1\varphi_0\left[\begin{matrix} a; & q, & z \\ - & & \end{matrix}\right] = \prod_{n=0}^{\infty}\frac{(1-azq^n)}{(1-zq^n)} = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{1.3.1}$$

known as Heine's theorem. The identity (1.3.1) is also known as $q-$binomial theorem because for limit $q \to 1$ it becomes

$$ {}_1F_0\left[\begin{matrix} a; & z \\ - & \end{matrix}\right] = (1-z)^{-a}, $$

which is an ordinary binomial series. To check the truth of (1.3.1), lets us consider the following two trivial identities

$$ {}_1\varphi_0\left[\begin{matrix} a; & q, & qz \\ - & & \end{matrix}\right] = {}_1\varphi_0\left[\begin{matrix} a; & q, & z \\ - & & \end{matrix}\right] - (1-a)\,{}_1\varphi_0\left[\begin{matrix} aq; & q, & z \\ - & & \end{matrix}\right] \tag{1.3.2}$$

and

$$ {}_1\varphi_0\left[\begin{matrix} a; & q, & z \\ - & & \end{matrix}\right] = (1-a)\,{}_1\varphi_0\left[\begin{matrix} aq; & q, & z \\ - & & \end{matrix}\right] + a\,{}_1\varphi_0\left[\begin{matrix} a; & q, & qz \\ - & & \end{matrix}\right]. \tag{1.3.3}$$

Eliminating

$$ {}_1\varphi_0 \left[\begin{matrix} aq; & q, & z \\ - & & \end{matrix} \right] $$

from (1.3.2) and (1.3.3), we get

$$ {}_1\varphi_0 \left[\begin{matrix} a; & q, & z \\ - & & \end{matrix} \right] = \frac{(1-az)}{(1-z)}\, {}_1\varphi_0 \left[\begin{matrix} a; & q, & qz \\ - & & \end{matrix} \right]. \tag{1.3.4} $$

Repeated use of (1.3.4) $(n-1)$ times yields

$$ {}_1\varphi_0 \left[\begin{matrix} a; & q, & z \\ - & & \end{matrix} \right] = \frac{(1-az)}{(1-z)} \frac{(1-azq)}{(1-zq)} \cdots\cdots \frac{(1-azq^{n-1})}{(1-zq^{n-1})} $$

$$ \times {}_1\varphi_0 \left[\begin{matrix} a; & q, & q^n z \\ - & & \end{matrix} \right], \tag{1.3.5} $$

which is for $|q| < 1$ and $n \to \infty$ gives (1.3.1).

Taking $a = 0$ in (1.3.1), we get

$$ {}_0\varphi_0 \left[\begin{matrix} -; & q, & z \\ - & & \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty} = e_q(z), $$

which is $q-$exponential function. If we replace z by z/a and take $a \to \infty$ in (1.3.1), we get

$$ \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} z^n}{(q;q)_n} = (z;q)_\infty = E_q(z). $$

The functions $e_q(z)$ and $E_q(z)$ are such that

$$e_q(z).E_q(z) = 1.$$

Also

$$ {}_1\varphi_0\left[\begin{array}{ccc} a; & q, & z \\ - & & \end{array}\right] {}_1\varphi_0\left[\begin{array}{ccc} b; & q, & az \\ - & & \end{array}\right] = {}_1\varphi_0\left[\begin{array}{ccc} ab; & q, & z \\ - & & \end{array}\right] \tag{1.3.6}$$

is the $q-$analogue of binomial addition theorem, *i.e.*

$$(1-z)^a(1-z)^b = (1-z)^{a+b}$$

The identity (1.3.1) is one of the most fundamental result in the theory of basic hypergeometric series. In fact a large number of summation and transformation identities of basic hypergeometric series can be derived by using (1.3.1). To demonstrate this fact we derive the following transformation formula

$$ {}_2\varphi_1\left[\begin{array}{cccc} a, & b; & q, & z \\ & c & & \end{array}\right] = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} \quad {}_2\varphi_1\left[\begin{array}{cccc} c/b, & z; & q, & b \\ & az & & \end{array}\right], \tag{1.3.7}$$

where $|z| < 1$ and $|b| < 1$. To prove (1.3.7), we set $a = c/b$ and $z = bq^n$ in (1.3.1) to obtain

$$\sum_{m=0}^{\infty} \frac{(c/b; q)_m}{(q; q)_m}(bq^n)^m = \frac{(cq^n; q)_\infty}{(bq^n; q)_\infty}. \tag{1.3.8}$$

We can write left side of (1.3.7) as

$$
{}_2\varphi_1\left[\begin{matrix} a, & b; & q, & z \\ & c & & \end{matrix}\right] = \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n(cq^n;q)_\infty}{(q;q)_n(bq^n;q)_\infty}z^n
$$

$$
= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n \sum_{m=0}^{\infty}\frac{(c/b;q)_m}{(q;q)_m}(bq^n)^m
$$

(by (1.3.8))

$$
= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_m}{(q;q)_m}b^m \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}(zq^m)^n
$$

$$
= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_m}{(q;q)_m}b^m \frac{(azq^m;q)_\infty}{(zq^m;q)_\infty},
$$

(by (1.3.1))

$$
= \frac{(b,az;q)_\infty}{(c,z;q)_\infty} \quad {}_2\varphi_1\left[\begin{matrix} c/b, & z; & q, & b \\ & az & & \end{matrix}\right].
$$

In a similar way we can also prove

$$
{}_2\varphi_1\left[\begin{matrix} a, & b; & q, & z \\ & c & & \end{matrix}\right] = \frac{(c/b,bz;q)_\infty}{(c,z;q)_\infty} \quad {}_2\varphi_1\left[\begin{matrix} abz/c, & b; & ;q, & c/b \\ & bz & & \end{matrix}\right]. \quad (1.3.9)
$$

and

$$
{}_2\varphi_1\left[\begin{matrix} a, & b; & q, & z \\ & c & & \end{matrix}\right] = \frac{(abz/c;q)_\infty}{(z;q)_\infty}\, {}_2\varphi_1\left[\begin{matrix} c/a, & c/b; & q, & abz/c \\ & c & & \end{matrix}\right]. \quad (1.3.10)
$$

The transformations (1.3.7), (1.3.9) and (1.3.10) are due to Heine [48, 49] and (1.3.10) assumed to be a $q-$analogue of *Euler's* transformation

$$ {}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right] = (1-z)^{c-a-b}\, {}_2F_1\left[\begin{matrix} c-a, & c-b; & z \\ & c & \end{matrix}\right] $$

If we set $z = c/ab$ in (1.3.7) and again use (1.3.1), we obtain

$$ {}_2\varphi_1\left[\begin{matrix} a, & b; & q, & c/ab \\ & c & & \end{matrix}\right] = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty}. \tag{1.3.11} $$

which is $q-$analogue of well known *Gauss* summation

$$ {}_2F_1\left[\begin{matrix} a, & b; & 1 \\ & c & \end{matrix}\right] = \frac{\Gamma c\, \Gamma(c-a-b)}{\Gamma(c-a)\, \Gamma(c-b)}. \tag{1.3.12} $$

Next using (1.3.1), we can write

$$ {}_1\varphi_0\left[\begin{matrix} a; & q, & z \\ - & & \end{matrix}\right] {}_1\varphi_0\left[\begin{matrix} 1/c; & q, & cz \\ - & & \end{matrix}\right] = \frac{(az;q)_\infty}{(cz;q)_\infty} $$

$$ = {}_1\varphi_0\left[\begin{matrix} a/c; & q, & cz \\ - & & \end{matrix}\right], \tag{1.3.13} $$

on equating the coefficients of z^n on both the sides of (1.3.13) and replacing c by cq^{n-1}, we get

$$ {}_2\varphi_1\left[\begin{matrix} a, & q^{-n}; & q, & q \\ & c & & \end{matrix}\right] = \frac{(c/a;q)_n a^n}{(c;q)_n} \tag{1.3.14} $$

which is $q-$analouge of *Vandermonde's* formula. If we reverse the order of summation in (1.3.14), we obtain

$$
{}_2\varphi_1\left[\begin{array}{cccc} a, & q^{-n}; & q, & cq^n/a \\ & c & & \end{array}\right] = \frac{(c/a;q)_n}{(c;q)_n}. \tag{1.3.15}
$$

i.e. terminating $q-$analouge of *Gauss's* summation formula. The summation (1.3.14) and (1.3.15) can also obtained directly by putting $b = q^{-n}$ in (1.3.11). The use of identities (1.3.14) and (1.3.15) with (1.3.1) also leads to the proof of the following transformations due to *Jackson*

$$
{}_2\varphi_1\left[\begin{array}{cccc} a, & b; & q, & z \\ & c & & \end{array}\right] = \frac{(az;q)_\infty}{(z;q)_\infty} \quad {}_2\varphi_2\left[\begin{array}{cccc} a, & c/b; & q, & bz \\ c, & az & & \end{array}\right]. \tag{1.3.16}
$$

and

$$
\begin{aligned}
{}_2\varphi_1\left[\begin{array}{cccc} a, & b; & q, & z \\ & c, & & \end{array}\right] &= \frac{(abz/c;q)_\infty}{(bz/c;q)_\infty} \quad {}_3\varphi_2\left[\begin{array}{ccccc} a, & c/b, & 0; & q; & q \\ & c, & cq/bz & & \end{array}\right] \\
&+ \frac{(a,bz,c/b;q)_\infty}{(c,z,c/bz;q)_\infty}\, {}_3\varphi_2\left[\begin{array}{ccccc} z, & abz/c, & 0; & q; & q \\ & bz, & bzq/c & & \end{array}\right],
\end{aligned} \tag{1.3.17}
$$

and (1.3.16) is $q-$analogue of *Pfaff-Kummer* transformation formula

$$
{}_2F_1\left[\begin{array}{ccc} a, & b; & z \\ & c & \end{array}\right] = (1-z)^{-a}\, {}_2F_1\left[\begin{array}{ccc} a, & c-b; & z/z-1 \\ & c & \end{array}\right]
$$

If we put $a = q^{-n}$ (1.3.16) and changing the summation index k by $n-k$, we obtain *Sear's* [76] transformation

$$
{}_2\varphi_1\left[\begin{matrix} q^{-n}, & b; & q; & z \\ & c & & \end{matrix}\right]
$$

$$
= \frac{(c/b;q)_n}{(c;q)_n}\left(\frac{bz}{q}\right)^n {}_3\varphi_2\left[\begin{matrix} q^{-n}, & q/z, & q^{1-n}/c; & q; & q \\ & bq^{1-n}/c, & 0 & & \end{matrix}\right]. \tag{1.3.18}
$$

Some more important results follow immediately by using (1.3.1) are *Bailey-Daum* (*q-Kummer*) summation formula

$$
{}_2\varphi_1\left[\begin{matrix} a, & b; & q, & -q/b \\ & aq/b & & \end{matrix}\right] = \frac{(a;q)_\infty(-q;q)_\infty(aq^2/b^2;q^2)_\infty}{(-q/b;q)_\infty(aq/b;q)_\infty(a;q^2)_\infty}, \tag{1.3.19}
$$

where $|q/b| < 1.$

q-Pfaff-Saalschütz sum

$$
{}_3\varphi_2\left[\begin{matrix} a, & b, & q^{-n}; & q, & q \\ & c, & abq^{1-n}/c & & \end{matrix}\right] = \frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n}. \tag{1.3.20}
$$

q-Dixon summation

$$
{}_4\varphi_3\left[\begin{matrix} a, & -qa^{1/2}, & b, & c; & q, & qa^{1/2}/bc \\ & -a^{1/2}, & aq/b, & aq/c & & \end{matrix}\right]
$$

$$
= \frac{(aq;q)_\infty(qa^{1/2}/b;q)_\infty(qa^{1/2}/c;q)_\infty(aq/bc;q)_\infty}{(aq/b;q)_\infty(aq/c;q)_\infty(qa^{1/2};q)_\infty(qa^{1/2}/bc;q)_\infty}. \tag{1.3.21}
$$

Watson's q-Whipple transformation

$$
{}_8\varphi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n}; & q, & q^{2+n}a^2/bcde \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{1+n} & & \end{matrix}\right]
$$

$$
= \frac{(aq;q)_n(aq/bc;q)_n}{(aq/b;q)_n(aq/c;q)_n}\,{}_4\varphi_3\left[\begin{matrix} b, & c, & q^{-n}, & aq/de; & q, & q \\ & aq/d, & aq/e, & bcq^{-n}/a & & \end{matrix}\right]. \quad (1.3.22)
$$

and *Jackson's q-Dougall-Dixon* formula

$$
{}_8\varphi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n}; & q, & q \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{1+n} & & \end{matrix}\right]
$$

$$
= \frac{(aq;q)_n(aq/bc;q)_n(aq/bd;q)_n(aq/cd;q)_n}{(aq/b;q)_n(aq/c;q)_n(aq/d;q)_n(aq/bcd;q)_n} \quad (1.3.23)
$$

where $q^{1+n}a^2 = bcde.$

A number of mathematicians have contributed fashionably in the development of the theory of basic hypergeometric functions, notably among them are Heine, Rogers, Ramanujan, Bailey, Slater, Watson, Dougall, Andrews, Jackson, Carlitz, Askey, Agarwal, Al-Salam, Gasper, Rahman, Ismail, Exton, Pathan, Srivastava, Saxena, Denis, Singh, Verma, Jain, Singh, Agarwal, Fine, Chen, Schlosser, Mc Laughlin and several others.

1.4 Bailey Lemma, Pairs and Chain

About 65 years ago, in 1944, Bailey wrote an influential paper [25] which was inspired by Rogers second proof of the Rogers-Ramanujan identities [71]. In his paper, Bailey gave a very trivial series identity which was later known as Bailey lemma which states that if

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}, \tag{1.4.1}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{n+r}, \tag{1.4.2}$$

then under the suitable convergence conditions and if change in the order of summations is allowed

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{1.4.3}$$

where $\alpha_r,\ \delta_r,\ u_r$ and v_r are functions of r such that β_n and γ_n exist. The proof is straightforward. In the left hand side of (1.4.3) using (1.4.2), we have

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \alpha_n \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}. \tag{1.4.4}$$

Assuming that this double series is absolutely convergent, we can interchange the order of summation to get

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{r=0}^{\infty} \sum_{n=0}^{r} \alpha_r \delta_r u_{r-n} v_{r+n}. \tag{1.4.5}$$

$$\sum_{n=0}^{\infty}\alpha_n\gamma_n = \sum_{r=0}^{\infty}\beta_r\delta_r \tag{1.4.6}$$

Bailey lemma is extended to a bilateral analogue in 1970 by Andrews [19] which is as follows. If

$$c_n = \sum_{m=0}^{\infty}a_{m+n}b_m, \tag{1.4.7}$$

then subject to suitable convergence conditions

$$\sum_{m=0}^{\infty}b_m \sum_{n=-\infty}^{\infty} a_n = \sum_{n=-\infty}^{\infty} c_n. \tag{1.4.8}$$

In (1.4.1) and (1.4.2), if we choose

$$u_r = \frac{1}{(q;q)_r}; \qquad v_r = \frac{1}{(aq;q)_r},$$

we get

$$\beta_n = \sum_{r=0}^{n}\frac{\alpha_r}{(q;q)_{n-r}\ (aq;q)_{n+r}}, \tag{1.4.9}$$

and

$$\gamma_n = \sum_{r=n}^{\infty}\frac{\delta_r}{(q;q)_{r-n}\ (aq;q)_{n+r}}. \tag{1.4.10}$$

A pair of sequences that satisfy (1.4.9) and (1.4.10) are respectively called Bailey pair and conjugate Bailey pair relative to a.

The concept of Bailey pair has been generalized in the works of Bressoud [28] and Singh [80]. But one of the most significant and elegant generalization is due to Andrews [17]. If we choose

$$u_r = \frac{(k/a;q)_r}{(q;q)_r}; \qquad v_r = \frac{(k;q)_r}{(aq;q)_r}$$

in (1.4.1), we obtain

$$\beta_n(a,k;q) = \sum_{r=0}^{n} \frac{(k/a;q)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}}\ \alpha_r(a,k;q). \tag{1.4.11}$$

Here, it is assumed that

$$\alpha_0(a,k;q) = 0.$$

The pair of sequence $(\alpha_n(a,k;q)\ \beta_n(a,k;q))$ that satisfy (1.4.11) is called WP-Bailey pair. WP (*i.e* Well-Poised) nature can verify by using (1.2.6) *i.e.*

$$\frac{(k/a;q)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}} = \frac{(k,k/a;q)_n}{(q,aq;q)_n}\frac{(kq^n,q^{-n};q)_r}{(aq^{1-n}/k,aq^{n+1};q)_r}\left(\frac{aq}{k}\right)^r.$$

For $k=0$ in (1.4.11), we get (1.4.9).

In fact Andrews [17] defined a process of constructing a new Bailey pairs from a known one. This has led to the concept of Bailey chain.

Recently, Liu and Ma [61], generalised the idea of WP-Bailey chain. An elliptic generalization of Andrews [17] has been given by Spiridonov [88]. A number of mathematicians including Warnaar [97, 98, 99, 100, 101], Schilling and Warnaar [82, 83, 84], Paul [63, 64, 65], Agarwal et al [6], Laughlin [58], Laughlin et al [59, 60], Rowell [72] and Bressoud at al [29] have contributed significantly in the development of Bailey pairs and chain. For further details of development and application of Bailey lemma one is referred to a recent survey by warnaar [97].

Chapter 2

Transformations of Basic and Poly-Basic Hypergeometric Series

2.1 Introduction

In 1944, Bailey [25] established a powerful series identity which was later known as Bailey's lemma. The Bailey's lemma states that, if

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}, \tag{2.1.1}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{n+r}, \tag{2.1.2}$$

The content of this chapter is based on the reference [7].

then, under the suitable convergence conditions and if change in the order of summations is allowable

$$\sum_{n=0}^{\infty}\alpha_n\gamma_n = \sum_{n=0}^{\infty}\beta_n\delta_n, \tag{2.1.3}$$

where α_r, δ_r, u_r and υ_r are functions of r such that γ_n exists.

The Bailey lemma has been a powerful tool in proving Rogers-Ramanujan type of identities and also a verity of transformations of basic and Poly-basic hypergeometric series. Slater [85, 86] used Bailey's lemma systematically to produced long list of 130 identities of Roger-Ramanujan type. Using the technique of Bailey lemma Denis [34], Denis and Singh [37], Laughlin [58] and Laughlin et al [59, 60] have established a number of identities of basic hypergeometric series.

In the present chapter, we have made an attempt to establish some interesting transformations and summations of basic and polybasic hypergeometric series by making use of Bailey lemma. Some special cases have also been mentioned.

In the development of this chapter, we shall require the following known results

$$ {}_5\varphi_4\left[\begin{matrix} a, & aq, & aq^2, & a^3q^{3n+3} & q^{-3n}; & q^3, & q^3 \\ & a^{3/2}q^{3/2}, & -a^{3/2}q^{3/2}, & a^{3/2}q^3, & -a^{3/2}q^3 & & \end{matrix}\right] $$

$$= \frac{(q^3;q^3)_n(aq;q)_n a^n}{(a^3q^3;q^3)_n(q;q)_n}. \tag{2.1.4}$$

[92]

$$
{}_4\varphi_3\left[\begin{array}{cccccc} x^2, & x^2q, & q^{-2n}, & b^2x^4y^2q^{2n+2}; & q^2, & q^2 \\ & bx^2q, & bx^2q^2, & x^4q^2 & & \end{array}\right]
$$

$$
= \frac{(-q;q)_n(bq;q)_n x^{2n}}{(-x^2q;q)_n(bx^2q;q)_n}. \quad (2.1.5)
$$

[92]

$$
{}_4\varphi_3\left[\begin{array}{cccccc} x, & -xq, & q^{-n}, & x^2y^2q^{n+1}; & q, & q \\ & xyq, & -xyq, & x^2q & & \end{array}\right]
$$

$$
= \frac{(q;q)_n(x^2q^2;q^2)_m(y^2q^2;q^2)_m x^n}{(x^2q;q)_n(x^2y^2q^2;q^2)_m(q^2;q^2)_m}, \quad (2.1.6)
$$

where m is greatest integer $\leq n/2$.

[92]

$$
{}_4\varphi_3\left[\begin{array}{cccccc} x, & -xq, & q^{-n}, & bx^2q^{n+2}; & q, & q \\ & xq\sqrt{b}, & -xq\sqrt{b}, & x^2q^2 & & \end{array}\right]
$$

$$
= \frac{(q;q)_n(bq^2x;q)_n(bx^2q^3;q^2)_m(bq^2;q^2)_m(xq^2;q)_{2m}x^n}{(xq;q)_n(bx^2q^2;q)_n(q^2;q^2)_m(x^2q^3;q^2)_m(bxq^2;q)_{2m}}, \quad (2.1.7)
$$

where m is greatest integer $\leq n/2$.

[92]

$$
{}_2\varphi_1\left[\begin{array}{cccc} a, & y; & q, & q \\ & ayq & & \end{array}\right]_n = \frac{(aq,yq;q)_n}{(q,ayq;q)_n}. \quad (2.1.8)
$$

[1]

$$
{}_4\varphi_3\left[\begin{matrix}\alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & e; & q, & 1/e\\ & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/e & & \end{matrix}\right]_n = \frac{(\alpha q, eq; q)_n}{(q, \alpha q/e; q)_n}\left(\frac{1}{e}\right)^n, \quad (2.1.9)
$$

where $|1/e| < 1$.

[1]

$$
{}_6\varphi_5\left[\begin{matrix}\alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & \beta, & \gamma, & \delta; & q, & q\\ & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/\beta, & \alpha q/\gamma, & \alpha q/\delta & & \end{matrix}\right]_n
$$

$$
= \frac{(\alpha q, \beta q, \gamma q, \delta q; q)_n}{(q, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta; q)_n}, \quad (2.1.10)
$$

where $\alpha = \beta\gamma\delta$.

([1], II.25)

$$
\sum_{i=0}^{n} \frac{(1-\alpha p^i q^i)(\alpha; p)_i(\beta; q)_i\beta^{-i}}{(1-\alpha)(q; q)_i(\alpha p/\beta; p)_i} = \frac{(\alpha p; p)_n(\beta q; q)_n\beta^{-n}}{(q; q)_n(\alpha p/\beta; p)_n}. \quad (2.1.11)
$$

([44], II.34)

$$
\sum_{i=0}^{n} \frac{(1-\alpha p^i q^i)(1-\beta p^i q^{-i})(\alpha, \beta; p)_i(\gamma, \alpha/\beta\gamma; q)_i q^i}{(1-\alpha)(1-\beta)(q, \alpha q/\beta; q)_i(\alpha p/\gamma, \beta\gamma p; p)_i}
$$

$$
= \frac{(\alpha p, \beta p; p)_n(\gamma q, \alpha q/\beta\gamma; q)_n}{(q, \alpha q/\beta; q)_n(\alpha p/\gamma, \beta\gamma p; p)_n}. \quad (2.1.12)
$$

([44], II.35)

$$
\sum_{r=0}^{n} \frac{(1-\alpha\delta p^r q^r)(1-\beta p^r/\delta q^r)(\alpha, \beta; p)_r(\gamma, \alpha\delta^2/\beta\gamma; q)_r}{(1-\alpha\delta)(1-\beta/\delta)(\delta q, \alpha\delta q/\beta; q)_r(\alpha\delta p/\gamma, \beta\gamma p/\delta; p)_r} q^r
$$

$$
= \frac{(1-\alpha)(1-\beta)(1-\gamma)(1-\alpha\delta^2/\beta\gamma)(\alpha p, \beta p; p)_n(\gamma q, \alpha\delta^2 q/\beta\gamma; q)_n}{\delta(1-\alpha\delta)(1-\beta/\delta)(1-\gamma/\delta)(1-\alpha\delta/\beta\gamma)(\delta q, \alpha\delta q/\beta; q)_n(\alpha\delta p/\gamma, \beta\gamma p/\delta; p)_n}
$$

$$- \frac{(1-\alpha)(1-\beta)(1-\gamma)(1-\alpha\delta^2/\beta\gamma)(\gamma/\alpha\delta, \delta/\beta\gamma; p)_1(1/\delta, \beta/\alpha\delta; q)_1}{\delta(1-\alpha\delta)(1-\beta/\delta)(1-\gamma/\delta)(1-\alpha\delta/\beta\gamma)(1/\gamma, \beta\gamma/\alpha\delta^2; q)_1(1/\alpha, 1/\beta; p)_1} \tag{2.1.13}$$

([44], II.36 for $m = 0$)

2.2 Transformations of Basic Hypergeometric Series

The following transformations are true whenever the series involved is convergent.

$$ {}_5\varphi_4 \left[\begin{matrix} a, & aq, & aq^2, & a^3q^{3n+3}, & q^{-3n}; & q^3, & zq^4 \\ & a^{3/2}q^{3/2}, & -a^{3/2}q^{3/2}, & a^{3/2}q^3, & -a^{3/2}q^3 & & \end{matrix} \right] $$

$$ = (1-zq)_3\varphi_2 \left[\begin{matrix} \omega q, & \omega^2 q, & q; & q, & azq \\ & aq\omega, & aq\omega^2 & & \end{matrix} \right], \tag{2.2.1}$$

where $\omega = e^{2\pi i/3}$ and $| zq^4 |< 1$, $| azq |< 1$.

$$ {}_4\varphi_3 \left[\begin{matrix} q^{-2n}, & b^2x^4y^2q^{2n+2}, & x^2, & x^2q; & q^2, & zq^3 \\ & bx^2q, & bx^2q^2, & x^4q^2 & & \end{matrix} \right] $$

$$ = (1-qz)_3\varphi_2 \left[\begin{matrix} q, & -q, & bq; & q, & x^2zq \\ & -x^2q, & bx^2q & & \end{matrix} \right], \tag{2.2.2}$$

where $| zq^3 |< 1$ and $| zx^2q |< 1$.

$$ {}_4\varphi_3 \left[\begin{matrix} x, & -xq, & x^2y^2q^{n+1}, & q^{-n}; & q, & zq^2 \\ & x^2q, & xyq, & -xyq & & \end{matrix} \right] $$

$$= (1-zq)\,{}_3\varphi_2\left[\begin{matrix} q, & y^2q^2, & q^2; & q^2, & x^2z^2q^2 \\ & x^2q, & x^2y^2q^2 & & \end{matrix}\right]$$

$$+ \frac{(1-zq)xzq}{(1-x^2q)}\;{}_3\varphi_2\left[\begin{matrix} q, & q^2, & y^2q^2; & q^2, & x^2z^2q^2 \\ & x^2q^3, & x^2y^2q^2 & & \end{matrix}\right], \quad (2.2.3)$$

where $\mid zq^2 \mid< 1$ and $\mid x^2z^2q^2 \mid< 1$.

$${}_4\varphi_3\left[\begin{matrix} q^{-n}, & bx^2q^{n+2}, & x, & -xq; & q, & zq^2 \\ & xq\sqrt{b}, & -xq\sqrt{b}, & x^2q^2 & & \end{matrix}\right]$$

$$= (1-zq)\,{}_4\varphi_3\left[\begin{matrix} q, & q^2, & bq^2, & xq^3; & q^2, & x^2z^2q^2 \\ & xq, & x^2q^3, & bx^2q^2 & & \end{matrix}\right]$$

$$+ \frac{xzq(1-zq)(1-bxq^2)}{(1-xq)(1-bx^2q^2)}\;{}_4\varphi_3\left[\begin{matrix} q, & q^2, & bq^2, & bxq^4; & q^2, & x^2z^2q^2 \\ & bx^2q^4, & x^2q^3, & bxq^2 & & \end{matrix}\right], \quad (2.2.4)$$

where $\mid zq^2 \mid< 1$ and $\mid x^2z^2q^2 \mid< 1$.

$${}_4\varphi_3\left[\begin{matrix} a, & y, & \alpha q, & eq; & q, & q/e \\ & q, & ayq, & \alpha q/e & & \end{matrix}\right]$$

$$+{}_6\varphi_5\left[\begin{matrix} aq, & yq, & \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & e; & q, & 1/e \\ & q, & ayq, & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/e & & \end{matrix}\right]$$

$$={}_6\varphi_5\left[\begin{matrix} a, & y, & \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & e; & q, & q/e \\ & q, & ayq, & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/e & & \end{matrix}\right], \quad (2.2.5)$$

for $|q/e| < 1$ and $|1/e| < 1$.

Choose $a = 0$ in (2.2.5), we get

$$
{}_3\varphi_2\left[\begin{matrix} y, & \alpha q, & eq; & q, & q/e \\ & q, & \alpha q/e & & \end{matrix}\right]
$$

$$
+{}_5\varphi_4\left[\begin{matrix} yq, & \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & e; & q, & 1/e \\ & q, & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/e & & \end{matrix}\right]
$$

$$
={}_5\varphi_4\left[\begin{matrix} y, & \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & e; & q, & q/e \\ & q, & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/e & & \end{matrix}\right], \quad (2.2.6)
$$

where $|q/e| < 1$ and $|1/e| < 1$.

$$
{}_6\varphi_5\left[\begin{matrix} a, & y, & \alpha q, & \beta q, & \gamma q, & \delta q; & q, & q \\ & q, & ayq, & \alpha q/\beta, & \alpha q/\gamma, & \alpha q/\delta & & \end{matrix}\right]
$$

$$
+{}_8\varphi_7\left[\begin{matrix} aq, & yq, & \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & \beta, & \gamma, & \delta; & q, & q \\ & q, & ayq, & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/\beta, & \alpha q/\gamma, & \alpha q/\delta & & \end{matrix}\right]
$$

$$
={}_8\varphi_7\left[\begin{matrix} a, & y, & \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & \beta, & \gamma, & \delta; & q, & q^2 \\ & q, & ayq, & \sqrt{\alpha}, & -\sqrt{\alpha}, & \alpha q/\beta, & \alpha q/\gamma, & \alpha q/\delta & & \end{matrix}\right]
$$

$$
+\frac{(aq, yq, \alpha q, \beta q, \gamma q, \delta q; q)_\infty}{(q, qyq, q, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta; q)_\infty}, \quad (2.2.7)
$$

where $|q| < 1$.

Proof of (2.2.1). Choosing $v_r = 1$, $u_r = q^r$ and

$$
\alpha_r = \frac{(a, aq, aq^2, a^3q^{3n+3}, q^{-3n}; q^3)_r q^{4r}}{(q^3, a^{3/2}q^{3/2}, -a^{3/2}q^{3/2}, a^{3/2}q^3, -a^{3/2}q^3; q^3)_r} \quad (2.2.8)
$$

in (2.1.1), we get

$$\beta_n = q^n \sum_{r=0}^{n} \frac{(a, aq, aq^2, a^3 q^{3n+3}, q^{-3n}; q^3)_r q^{3r}}{(q^3, a^{3/2} q^{3/2}, -a^{3/2} q^{3/2}, a^{3/2} q^3, -a^{3/2} q^3; q^3)_r}.$$

By using (2.1.4), we get

$$\beta_n = \frac{(q^3; q^3)_n (aq; q)_n (aq)^n}{(a^3 q^3; q^3)_n (q; q)_n}, \tag{2.2.9}$$

setting

$$\delta_r = z^r \tag{2.2.10}$$

in (2.1.2), we get

$$\gamma_n = \frac{z^n}{(1 - zq)}, \tag{2.2.11}$$

where $\mid zq \mid < 1$, using (2.2.8), (2.2.9), (2.2.10) and (2.2.11) in (2.1.3), we get (2.2.1).

Proof of (2.2.2). Choosing $v_r = 1$, $u_r = q^r$ and

$$\alpha_r = \frac{(b^2 x^4 y^2 q^{2n+2}, x^2, x^2 q, q^{-2n}; q^2)_r q^{3r}}{(q^2, bx^2 q, bx^2 q^2, x^4 q^2; q^2)_r} \tag{2.2.12}$$

in (2.1.1), we get

$$\beta_n = q^n \sum_{r=0}^{n} \frac{(b^2 x^4 y^2 q^{2n+2}, x^2, x^2 q, q^{-2n}; q^2)_r q^{2r}}{(q^2, bx^2 q, bx^2 q^2, x^4 q^2; q^2)_r},$$

by using (2.1.5), we obtain

$$\beta_n = \frac{(-q; q)_n (bq; q)_n (qx^2)^n}{(-x^2 q; q)_n (bx^2 q; q)_n}. \tag{2.2.13}$$

By making the use of (2.2.10), (2.2.11), (2.2.12) and (2.2.13) in (2.1.3), we obtain (2.2.2).

Proof of (2.2.3). Choosing $v_r = 1$, $u_r = q^r$ and

$$\alpha_r = \frac{(x^2y^2q^{n+1}, x, -xq, q^{-n}; q)_r q^{2r}}{(q, x^2q, xyq, -xyq; q)_r} \tag{2.2.14}$$

in (2.1.1), we get

$$\beta_n = q^n \sum_{r=0}^{n} \frac{(x^2y^2q^{n+1}, x, -xq, q^{-n}; q)_r q^r}{(q, x^2q, xyq, -xyq; q)_r},$$

by using (2.1.6), we get

$$\beta_n = \frac{(q;q)_n (x^2q^2;q^2)_m (y^2q^2;q^2)_m (xq)^n}{(x^2q;q)_n (x^2y^2q^2;q^2)_m (q^2;q^2)_m}, \tag{2.2.15}$$

where m is the greatest integer $\leq n/2$.

Now using (2.2.10), (2.2.11), (2.2.14) and (2.2.15) in (2.1.3), we get (2.2.3).

Proof of (2.2.4). Choosing $v_r = 1$, $u_r = q^r$ and

$$\alpha_r = \frac{(bx^2q^{n+2}, x, -xq, q^{-n}; q)_r q^{2r}}{(q, xq\sqrt{b}, -xq\sqrt{b}, x^2q^2; q)_r} \tag{2.2.16}$$

in (2.1.1), we get

$$\beta_n = q^n \sum_{r=0}^{n} \frac{(bx^2q^{n+2}, x, -xq, q^{-n}; q)_r q^r}{(q, xq\sqrt{b}, -xq\sqrt{b}, x^2q^2; q)_r},$$

by using (2.1.7), we get

$$\beta_n = \frac{(q;q)_n(bxq^2;q)_n(bx^2q^3;q^2)_m(bq^2;q^2)_m(xq^2;q)_{2m}(xq)^n}{(xq;q)_n(bx^2q^2;q)_n(q^2;q^2)_m(x^2q^3;q^2)_m(bxq^2;q)_{2m}}, \tag{2.2.17}$$

where m is the greatest integer $\leq n/2$.

By using (2.2.10), (2.2.11), (2.2.16) and (2.2.17) in (2.1.3), we obtain (2.2.4).

Proof of (2.2.5). let us choose $u_r = v_r = 1$ in (2.1.1) and (2.1.2), we get

$$\beta_n = \sum_{r=0}^{n} \alpha_r, \tag{2.2.18}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r,$$

and γ_r can be written as,

$$\gamma_n = \sum_{r=0}^{\infty} \delta_r - \sum_{r=0}^{n} \delta_r + \delta_n. \tag{2.2.19}$$

Choose

$$\delta_n = \frac{(a,y;q)_n q^n}{(q,ayq;q)_n} \tag{2.2.20}$$

in (2.2.19), gives

$$\gamma_n = \sum_{r=0}^{\infty} \frac{(a,y;q)_r q^r}{(q,ayq;q)_r} - \sum_{r=0}^{n} \frac{(a,y;q)_r q^r}{(q,ayq;q)_r} + \frac{(a,y;q)_n q^n}{(q,ayq;q)_n},$$

by using (2.1.8), we get

$$\gamma_n = \frac{(aq,yq;q)_\infty}{(q,ayq;q)_\infty} - \frac{(aq,yq;q)_n}{(q,ayq;q)_n} + \frac{(a,y;q)_n q^n}{(q,ayq;q)_n}, \tag{2.2.21}$$

substituting (2.2.20) and (2.2.21) in (2.1.3), we obtain the following relation

$$\sum_{n=0}^{\infty}\alpha_n\left[\frac{(aq,yq;q)_\infty}{(q,ayq;q)_\infty}-\frac{(aq,yq;q)_n}{(q,ayq;q)_n}+\frac{(a,y;q)_nq^n}{(q,ayq;q)_n}\right]$$

$$=\sum_{n=0}^{\infty}\beta_n\frac{(a,y;q)_nq^n}{(q,ayq;q)_n}. \qquad (2.2.22)$$

By choosing

$$\alpha_r=\frac{(\alpha,q\sqrt{\alpha},-q\sqrt{\alpha},e;q)_r}{(q,\sqrt{\alpha},-\sqrt{\alpha},\alpha q/e;q)_r}\left(\frac{1}{e}\right)^r \qquad (2.2.23)$$

in (2.2.18), we get

$$\beta_n=\sum_{r=0}^{n}\frac{(\alpha,q\sqrt{\alpha},-q\sqrt{\alpha},e;q)_r}{(q,\sqrt{\alpha},-\sqrt{\alpha},\alpha q/e;q)_r}\left(\frac{1}{e}\right)^r,$$

by using (2.1.9), we get

$$\beta_n=\frac{(\alpha q,eq;q)_n}{(q,\alpha q/e;q)_n}\left(\frac{1}{e}\right)^n, \qquad (2.2.24)$$

now put α_n and β_n in (2.2.22), we have (2.2.5).

Proof of (2.2.7). Substituting

$$\alpha_r=\frac{(\alpha,q\sqrt{\alpha},-q\sqrt{\alpha},\beta,\gamma,\delta;q)_rq^r}{(q,\sqrt{\alpha},-\sqrt{\alpha},\alpha q/\beta,\alpha q/\gamma,\alpha q/\delta;q)_r} \qquad (2.2.25)$$

in (2.2.18), we get

$$\beta_r=\sum_{r=0}^{n}\frac{(\alpha,q\sqrt{\alpha},-q\sqrt{\alpha},\beta,\gamma,\delta;q)_rq^r}{(q,\sqrt{\alpha},-\sqrt{\alpha},\alpha q/\beta,\alpha q/\gamma,\alpha q/\delta;q)_r},$$

on employing (2.1.10), we have

$$\beta_n = \frac{(\alpha q, \beta q, \gamma q, \delta q; q)_n}{(q, \alpha q/\beta, \alpha q/\gamma, \alpha q/\delta; q)_n}. \tag{2.2.26}$$

By substituting α_n and β_n in (2.2.22), we get (2.2.7).

2.3 Transformations of Poly-Basic Hypergeometric Series

The following transformations are true whenever the series involved is convergent

$$\varphi\begin{bmatrix} a, & y, & \beta q & : \alpha p; & q, & p; & q/\beta \\ q, & ayq & : \alpha p/\beta & & & & \end{bmatrix}$$
$$+\varphi\begin{bmatrix} aq, & yq, & \beta & : \alpha & : \alpha pq; & q, & p, & pq; & 1/\beta \\ q, & ayq & : \alpha p/\beta & : \alpha & & & & & \end{bmatrix}$$
$$= \varphi\begin{bmatrix} a, & y, & \beta : & \alpha & : & \alpha pq; & q, & p, & pq; & q/\beta \\ q, & ayq & : & \alpha p/\beta & : \alpha & & & & & \end{bmatrix} \tag{2.3.1}$$

which converges for max $(|1/\beta|, |q/\beta|) < 1$.

Taking $p = q$ in (2.3.1), we get

$${}_4\varphi_3\begin{bmatrix} a, & y, & \beta q, & \alpha q; & q, & q/\beta \\ & q, & ayq, & \alpha q/\beta & & \end{bmatrix}$$

$$+\,\varphi\left[\begin{array}{llll} aq, & yq, & \beta, & \alpha:\ \alpha q^2;\ \ q,\ \ q^2;\ \ 1/\beta \\ q, & ayq, & \alpha q/\beta & :\ \ \alpha \end{array}\right]$$

$$=\varphi\left[\begin{array}{llll} a, & y, & \beta, & \alpha:\ \alpha q^2;\ \ q,\ \ q^2;\ \ q/\beta \\ q, & ayq, & \alpha q/\beta: & \alpha \end{array}\right], \quad (2.3.2)$$

which converges for max $(|1/\beta|, |q/\beta|) < 1$.

$$\varphi\left[\begin{array}{llllllll} aq, & yq, & \gamma, & \alpha/\beta\gamma & :\alpha, & \beta & :\alpha pq & :\beta p/q \\ q, & ayq, & \alpha q/\beta & :\alpha p/\gamma, & \beta\gamma p & :\alpha & :\beta & \end{array}\right.$$

$$\left.\begin{array}{l} ;\ \ q,\ \ p,\ \ pq,\ \ p/q;\ \ q \\ \ \end{array}\right]$$

$$+\,\varphi\left[\begin{array}{lllll} a, & y, & \gamma q, & \alpha q/\beta\gamma & :\alpha p,\ \ \beta p;\ \ q,\ \ p;\ \ q \\ q, & ayq, & \alpha q/\beta & :\alpha p/\gamma, & \beta\gamma p \end{array}\right]$$

$$=\frac{(aq, yq, \gamma q, \alpha q/\beta\gamma; q)_\infty(\alpha p, \beta p; p)_\infty}{(q, q, ayq, \alpha q/\beta; q)_\infty(\alpha p/\gamma, \beta\gamma p; p)_\infty}$$

$$+\,\varphi\left[\begin{array}{llllllll} a, & y, & \gamma, & \alpha/\beta\gamma & :\alpha, & \beta & :\alpha pq & :\beta p/q \\ q, & ayq, & \alpha q/\beta & :\alpha p/\gamma, & \beta\gamma p & :\alpha & :\beta & \end{array}\right.$$

$$\left.\begin{array}{l} ;\ \ q\ \ ,p\ \ ,pq\ \ ,p/q;\ \ q^2 \\ \ \end{array}\right]. \quad (2.3.3)$$

Changing $p \to q^2$ in (2.3.3), we get

$$\varphi\left[\begin{array}{lllllll} aq, & yq, & \gamma, & \alpha/\beta\gamma, & \beta q: & \alpha, & \beta\ \ :\alpha q^3;\ \ q,\ \ q^2,\ \ q^3;\ \ q \\ q, & ayq, & \alpha q/\beta, & \beta: & \alpha q^2/\gamma, & \beta\gamma q^2 & :\alpha \end{array}\right]$$

$$+\,\varphi\left[\begin{array}{lllll} a, & y, & \gamma q, & \alpha q/\beta\gamma: & \alpha q^2,\ \ \beta q^2;\ \ q,\ \ q^2;\ \ q \\ q, & ayq, & \alpha q/\beta: & \alpha q^2/\gamma, & \beta\gamma q^2 \end{array}\right]$$

$$= \frac{(aq, yq, \gamma q, \alpha q/\beta\gamma; q)_\infty (\alpha q^2, \beta q^2; q^2)_\infty}{(q, q, ayq, \alpha q/\beta; q)_\infty (\alpha q^2/\gamma, \beta\gamma q^2; q^2)_\infty}$$

$$+\varphi\left[\begin{array}{llllllllllll} a, & y, & \gamma, & \alpha/\beta\gamma, & \beta q: & \alpha, & \beta & :\alpha q^3; & q, & q^2, & q^3; & q^2 \\ q, & ayq, & \alpha q/\beta, & \beta: & \alpha q^2/\gamma, & \beta\gamma q^2 & :\alpha & & & & & \end{array}\right]. \tag{2.3.4}$$

$$\frac{(1-\alpha)(1-\beta)(1-\gamma)(1-\alpha\delta^2/\beta\gamma)}{\delta(1-\alpha\delta)(1-\beta/\delta)(1-\gamma/\delta)(1-\alpha\delta/\beta\gamma)}$$

$$\times\left\{\varphi\left[\begin{array}{llllllll} a, & y, & \gamma q, & \alpha\delta^2 q/\beta\gamma & :\alpha p, & \beta p; & q, & p; \quad q \\ ayq, & \delta q, & \alpha\delta q/\beta & :\alpha\delta p/\gamma, & \beta\gamma p/\delta & & & \end{array}\right]\right.$$

$$\left. - \frac{(aq, yq; q)_\infty}{(q, ayq; q)_\infty} \frac{(1-\gamma/\alpha\delta)(1-\delta/\beta\gamma)(1-1/\delta)(1-\beta/\alpha\delta)}{(1-1/\gamma)(1-\beta\gamma/\alpha\delta^2)(1-1/\alpha)(1-1/\beta)}\right\}$$

$$= \frac{(aq, yq, \gamma q, \alpha\delta^2 q/\gamma; q)_\infty (\alpha p, \beta p; p)_\infty}{(q, ayq, \delta q, \alpha\delta q/\beta; q)_\infty (\alpha\delta p/\gamma, \beta\gamma p/\delta; p)_\infty} \frac{(1-\alpha)(1-\beta)(1-\gamma)}{\delta(1-\alpha\delta)(1-\beta/\delta)}$$

$$\times \frac{(1-\alpha\delta^2/\beta\gamma)}{(1-\gamma/\delta)(1-\alpha\delta/\beta\gamma)} - \frac{(aq, yq; q)_\infty}{(q, ayq; q)_\infty} \frac{(1-\gamma/\alpha\delta)(1-\delta/\beta\gamma)}{(1-1/\alpha)(1-1/\beta)}$$

$$\times \frac{(1-1/\delta)(1-\beta/\alpha\delta)}{(1-1/\gamma)(1-\beta\gamma/\alpha\delta^2)} \frac{(1-\alpha)(1-\beta)(1-\gamma)(1-\alpha\delta^2/\beta\gamma)}{\delta(1-\alpha\delta)(1-\beta/\delta)(1-\gamma/\delta)(1-\alpha\delta/\beta\gamma)}$$

$$-\varphi\left[\begin{array}{llllllll} aq, & yq, & \gamma, & \alpha\delta^2/\beta\gamma & :\alpha, & \beta & :\alpha\delta pq & :\beta p/\delta q \\ ayq, & \delta q, & \alpha\delta q/\beta & :\alpha\delta p/\gamma, & \beta\gamma p/\delta & :\alpha\delta & :\beta/\delta & \end{array}\right.$$

$$\left. ; \quad q, \quad p, \quad pq, \quad p/q; \quad q \right]$$

$$+\varphi\left[\begin{array}{llllllll} a, & y, & \gamma, & \alpha\delta^2/\beta\gamma & :\alpha, & \beta & :\alpha\delta pq & :\beta p/\delta q \\ ayq, & \delta q, & \alpha\delta q/\beta & :\alpha\delta p/\gamma, & \beta\gamma p/\delta & :\alpha\delta & :\beta/\delta & \end{array}\right.$$

$$; \quad q, \quad p, \quad pq, \quad p/q; \quad q^2 \Bigg]. \tag{2.3.5}$$

Proof of (2.3.1). By taking

$$\alpha_r = \frac{(\alpha pq; pq)_r (\alpha; p)_r (\beta; q)_r \beta^{-r}}{(\alpha; pq)_r (\alpha p/\beta; p)_r (q; q)_r} \tag{2.3.6}$$

in (2.2.18), we have

$$\beta_n = \sum_{r=0}^{n} \frac{(\alpha pq; pq)_r (\alpha; p)_r (\beta; q)_r \beta^{-r}}{(\alpha; pq)_r (\alpha p/\beta; p)_r (q; q)_r},$$

by using (2.1.11), we obtain

$$\beta_n = \frac{(\alpha p; p)_n (\beta q; q)_n \beta^{-n}}{(q; q)_n (\alpha p/\beta; p)_n}, \tag{2.3.7}$$

Now put α_n and β_n in (2.2.22), we obtain (2.3.1).

Proof of (2.3.3). Let us choose

$$\alpha_r = \frac{(\alpha pq; pq)_r (\beta p/q; p/q)_r (\alpha, \beta; p)_r (\gamma, \alpha/\beta\gamma; q)_r q^r}{(\alpha; pq)_r (\beta; p/q)_r (q, \alpha q/\beta; q)_r (\alpha p/\gamma, \beta\gamma p; p)_r} \tag{2.3.8}$$

in (2.2.18), we have

$$\beta_n = \sum_{r=0}^{n} \frac{(\alpha pq; pq)_r (\beta p/q; p/q)_r (\alpha, \beta; p)_r (\gamma, \alpha/\beta\gamma; q)_r q^r}{(\alpha; pq)_r (\beta; p/q)_r (q, \alpha q/\beta; q)_r (\alpha p/\gamma, \beta\gamma p; p)_r},$$

on employing (2.1.12), we get

$$\beta_n = \frac{(\alpha p, \beta p; p)_n(\gamma q, \alpha q/\beta\gamma; q)_n}{(q, \alpha q/\beta; q)_n(\alpha p/\gamma, \beta\gamma p; p)_n}. \tag{2.3.9}$$

Substituting α_n and β_n in (2.2.22), we get (2.3.3).

Proof of (2.3.5). Let us choose

$$\alpha_r = \frac{(\alpha\delta pq; pq)_r(\beta p/\delta q; p/q)_r(\alpha, \beta; p)_r(\gamma, \alpha\delta^2/\beta\gamma; q)_r q^r}{(\alpha\delta; pq)_r(\beta/\delta; p/q)_r(\alpha\delta p/\gamma, \beta\gamma p/\delta; p)_r(\delta q, \alpha\delta q/\beta; q)_r}, \tag{2.3.10}$$

in (2.2.18) and employing (2.1.13), we have

$$\begin{aligned}\beta_n = {} & \frac{(1-\alpha)(1-\beta)(1-\gamma)(1-\alpha\delta^2/\beta\gamma)}{\delta(1-\alpha\delta)(1-\beta/\delta)(1-\gamma/\delta)(1-\alpha\delta/\beta\gamma)} \\ & \qquad \times \frac{(\alpha p, \beta p; p)_n(\gamma q, \alpha\delta^2 q/\beta\gamma; q)_n}{(\delta q, \alpha\delta q/\beta; q)_n(\alpha\delta p/\gamma, \beta\gamma p/\delta; p)_n} \\ & - \frac{(1-\alpha)(1-\beta)(1-\gamma)(1-\alpha\delta^2/\beta\gamma)}{\delta(1-\alpha\delta)(1-\beta/\delta)(1-\gamma/\delta)(1-\alpha\delta/\beta\gamma)} \\ & \qquad \times \frac{(\gamma/\alpha\delta, \delta/\beta\gamma; p)_1(1/\delta, \beta/\alpha\delta; q)_1}{(1/\gamma, \beta\gamma/\alpha\delta^2; q)_1(1/\alpha, 1/\beta; p)_1}.\end{aligned} \tag{2.3.11}$$

By using α_n and β_n in (2.2.22), we get (2.3.5).

Chapter 3

Some New WP-Bailey Pairs and Basic Hypergeometric Series Identities

.

3.1 Introduction

As discussed in the previous chapter, the Bailey lemma has been a powerful tool in the discovery of identities of Rogers-Ramanujan type and also a variety of ordinary and basic hypergeometric series identities. In fact, during last seven decades Bailey lemma and its various generalizations have proved to be a powerful tool in the discoveries of Rogers-Ramanujan type of identities, transformations and summations theorems of ordinary and basic hypergeometric series. Slater [85, 86] used Bailey lemma to discover the famous list

The content of this chapter is based on the reference [8] of 130 identities of Rogers-Ramanujan type. Using the same tool more identities of Rogers-Ramanujan type and basic hypergeometric series have been given by Andrew [16], Foda and Quano [43], Denis [35], Denis and Singh [37] and Singh [79].

In (2.1.1), if we choose $u_r = 1/(q;q)_r$ and $v_r = 1/(aq;q)_r$, we get

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}. \tag{3.1.1}$$

The pair of sequence $(\alpha_n,\ \beta_n)$ that satisfies (3.1.1) is called a Bailey pair relative to the parameter a. The concept of Bailey pairs has been generalized in the works of Bressoud [28] and Singh [80]. The most elegant generalizations of Bailey pair have been given by Andrews [17] which is

$$\beta_n = \sum_{r=0}^{n} \frac{(k/a;q)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}} \alpha_r. \tag{3.1.2}$$

The pair (α_n, β_n) satisfying (3.1.2) is termed as WP-Bailey pair. It is easy to see that (3.1.2) follows by setting $u_r = \frac{(k/a;q)_r}{(q;q)_r}$ and $v_r = \frac{(k;q)_r}{(aq;q)_r}$ in (2.1.1). For $k = 0$ in (3.1.2), we get the standard Bailey pair (3.1.1).

Andrew et al [13, 14, 15, 16] have exploited very effective the mechanism of Bailey lemma in the form of Bailey chain. Infact, Andrews [17] described that the process may be iterated to produce a chain of WP-Bailey pairs constructing new WP-bailey pairs from existing initial WP-Bailey pair and introduced the concept of Bailey chain. The aforesaid technique developed by Andrews have been effectively used to discover new WP-Bailey pairs. The idea of WP-Bailey pairs and chain has further been generalized by Liu and

Ma [61] and Spiridonov [88].

In the present chapter, we have established some new WP-Bailey pairs using the following results of Andrews [17] and Warnaar [98]. We, further, have used the new WP-Bailey pairs to produced a number of new transformations of basic hypergeometric series.

Theorem 1 [17]. If $(\alpha_n(a,k;q),\ \beta_n(a,k;q))$ is a WP-Bailey pair and satisfies (3.1.2), then so is the pair $(\alpha_n^{'}(a,k;q),\ \beta_n^{'}(a,k;q))$ given by

$$\alpha_n^{'}(a,k;q) = \frac{(\rho_1,\rho_2;q)_n}{(aq/\rho_1,aq/\rho_2;q)_n}\left(\frac{k}{m}\right)^n \alpha_n(a,m;q). \tag{3.1.3}$$

$$\beta_n^{'}(a,k;q) = \frac{(mq/\rho_1,mq/\rho_2;q)_n}{(aq/\rho_1,aq/\rho_2;q)_n}\sum_{r=0}^{n}\frac{(1-mq^{2r})(\rho_1,\rho_2;q)_r(k/m;q)_{n-r}}{(1-m)(mq/\rho_1,mq/\rho_2;q)_r(q;q)_{n-r}} \times\frac{(k;q)_{n+r}}{(mq;q)_{n+r}}\left(\frac{k}{m}\right)^r\beta_r(a,m;q), \tag{3.1.4}$$

where $m = k\rho_1\rho_2/aq$.

Theorem 2 [17]. If $(\alpha_n(a,k;q),\ \beta_n(a,k;q))$ is a WP-Bailey pair, then so is the pair $(\alpha_n^{'}(a,k;q),\ \beta_n^{'}(a,k;q))$ given by

$$\alpha_n^{'}(a,k;q) = \frac{(m;q)_{2n}}{(k;q)_{2n}}\left(\frac{k}{m}\right)^n \alpha_n(a,m;q). \tag{3.1.5}$$

$$\beta_n^{'}(a,k;q) = \sum_{r=0}^{n}\frac{(k/m;q)_{n-r}}{(q;q)_{n-r}}\left(\frac{k}{m}\right)^r\beta_r(a,m;q), \tag{3.1.6}$$

where $m = a^2q/k$.

Theorem 3 [17]. If $(\alpha_n(a,k;q),\ \beta_n(a,k;q))$ is a WP-Bailey pair, then so is the pair $(\alpha_n^{'}(a,k;q),\ \beta_n^{'}(a,k;q))$ given by

$$\alpha_n^{'}(a,k;q) = \frac{(qa^2/k;q)_{2n}}{(k;q)_{2n}}\left(\frac{k^2}{qa^2}\right)^n \alpha_n(a,qa^2/k;q). \tag{3.1.7}$$

$$\beta_n^{'}(a,k;q) = \sum_{r=0}^{n}\frac{(k^2/qa^2;q)_{n-r}}{(q;q)_{n-r}}\left(\frac{k^2}{qa^2}\right)^r \beta_r(a,qa^2/k;q). \tag{3.1.8}$$

Theorem 4 [98]. If $(\alpha_n(a,k;q),\ \beta_n(a,k;q))$ is a WP-Bailey pair, then so is the pair $(\alpha_n^{'}(a,k;q),\ \beta_n^{'}(a,k;q))$ given by

$$\alpha_n^{'}(a,k;q) = \frac{(1-\sigma\sqrt{k})(1+\sigma\sqrt{mq^n})(m;q)_{2n}}{(1-\sigma\sqrt{kq^n})(1+\sigma\sqrt{m})(k;q)_{2n}}\left(\frac{k}{m}\right)^n \alpha_n(a,m;q). \tag{3.1.9}$$

$$\beta_n^{'}(a,k;q) = \frac{(1-\sigma\sqrt{k})}{(1-\sigma\sqrt{kq^n})}\sum_{r=0}^{n}\frac{(1+\sigma\sqrt{mq^r})(k/m;q)_{n-r}}{(1+\sigma\sqrt{m})(q;q)_{n-r}}\left(\frac{k}{m}\right)^r \beta_r(a,m;q), \tag{3.1.10}$$

where $m = a^2/k$ and $\sigma \in (-1,1)$.

Theorem 5 [98]. If $(\alpha_n(a,k;q),\ \beta_n(a,k;q))$ is a WP-Bailey pair, then so is the pair $(\alpha_n^{'}(a,k;q),\ \beta_n^{'}(a,k;q))$ given by

$$\alpha_n^{'}(a^2,k;q^2) = \alpha_n(a,m;q). \tag{3.1.11}$$

$$\beta_n^{'}(a^2,k;q^2) = \frac{(-mq;q)_{2n}}{(-aq;q)_{2n}}\sum_{r=0}^{n}\frac{(1-mq^{2r})(k/m^2;q^2)_{n-r}}{(1-m)(q^2;q^2)_{n-r}} \times\frac{(k;q^2)_{n+r}}{(m^2q^2;q^2)_{n+r}}\left(\frac{m}{a}\right)^{n-r}\beta_r(a,m;q), \tag{3.1.12}$$

where $m = k/aq$.

Theorem 6 [98]. If $(\alpha_n(a,k;q),\ \beta_n(a,k;q))$ is a WP-Bailey pair, then so is the pair $(\alpha_n^{'}(a,k;q),\ \beta_n^{'}(a,k;q))$ given by

$$\alpha_n^{'}(a^2,k;q^2) = q^{-n}\frac{(1+aq^{2n})}{(1+a)}\alpha_n(a,m;q). \tag{3.1.13}$$

$$\beta_n^{'}(a^2,k;q^2) = \frac{(-mq;q)_{2n}q^{-n}}{(-a;q)_{2n}}\sum_{r=0}^{n}\frac{(1-mq^{2r})(k/m^2;q^2)_{n-r}}{(1-m)(q^2;q^2)_{n-r}} \times\frac{(k;q^2)_{n+r}}{(m^2q^2;q^2)_{n+r}}\left(\frac{m}{a}\right)^{n-r}\beta_r(a,m;q), \tag{3.1.14}$$

where $m = k/a$.

Theorem 7 [98]. If $(\alpha_n(a,k;q),\ \beta_n(a,k;q))$ is a WP-Bailey pair, then so is the pair $(\alpha_n^{'}(a,k;q),\ \beta_n^{'}(a,k;q))$ given by

$$\alpha_{2n}^{'}(a,k;q) = \alpha_n(a,m;q^2) \quad ; \qquad \alpha_{2n+1}^{'}(a,k;q) = 0. \tag{3.1.15}$$

$$\beta_n^{'}(a,k;q) = \frac{(mq;q^2)_n}{(aq;q^2)_n}\sum_{r=0}^{[n/2]}\frac{(1-mq^{4r})(k/m;q)_{n-2r}}{(1-m)(q;q)_{n-2r}} \times\frac{(k;q)_{n+2r}}{(mq;q)_{n+2r}}\left(\frac{-k}{a}\right)^{n-2r}\beta_r(a,m;q^2), \tag{3.1.16}$$

where $m = k^2/a$.

In the next section, we shall also require the following identities.

$${}_4\varphi_3\left[\begin{matrix} a, & -q\sqrt{a}, & b, & q^{-n}; & q, & q^{n+1}a^{1/2}/b \\ & -\sqrt{a}, & aq/b, & aq^{n+1} & & \end{matrix}\right] = \frac{(aq, q\sqrt{a}/b;q)_n}{(q\sqrt{a}, aq/b;q)_n}. \tag{3.1.17}$$

([44]; II.14)

For $b = kq^n$ in (3.1.17), we get

$$
{}_4\varphi_3\left[\begin{matrix} a, & -q\sqrt{a}, & kq^n, & q^{-n}; & q, & qa^{1/2}/k \\ & -\sqrt{a}, & aq^{1-n}/k, & aq^{n+1} & & \end{matrix}\right]
= \frac{(aq, k/\sqrt{a}; q)_n}{(q\sqrt{a}, k/a; q)_n}\left(\frac{1}{\sqrt{a}}\right)^n. \quad (3.1.18)
$$

$$
{}_4\varphi_3\left[\begin{matrix} a, & c, & aq^{n+1/2}/c, & q^{-n}; & q, & q^2 \\ & aq/c, & cq^{1/2-n}, & aq^{n+1} & & \end{matrix}\right]
= \frac{(1+\sqrt{a})(aq, \sqrt{q}, \sqrt{aq}/c, q\sqrt{a}/c; q)_n}{(2\sqrt{a})(aq/c, \sqrt{q}/c, \sqrt{aq}, q\sqrt{a}; q)_n}
- \frac{(1-\sqrt{a})(aq, \sqrt{q}, \sqrt{aq}/c, -q\sqrt{a}/c; q)_n}{(2\sqrt{a})(aq/c, -\sqrt{q}/c, \sqrt{aq}, -q\sqrt{a}; q)_n}. \quad (3.1.19)
$$

([93]; 44)

For $c = a\sqrt{q}/k$ in (3.1.19), we get

$$
{}_4\varphi_3\left[\begin{matrix} a, & a\sqrt{q}/k, & kq^n, & q^{-n}; & q, & q^2 \\ & k\sqrt{q}, & aq^{1-n}/k, & aq^{n+1} & & \end{matrix}\right]
= \frac{(1+\sqrt{a})(aq, \sqrt{q}, k/\sqrt{a}, k\sqrt{q}/\sqrt{a}; q)_n}{(2\sqrt{a})(k\sqrt{q}, k/a, \sqrt{aq}, q\sqrt{a}; q)_n}
- \frac{(1-\sqrt{a})(aq, \sqrt{q}, k/\sqrt{a}, -k\sqrt{q}/\sqrt{a}; q)_n}{(2\sqrt{a})(k\sqrt{q}, -k/a, \sqrt{aq}, -q\sqrt{a}; q)_n}. \quad (3.1.20)
$$

3.2 New WP-Bailey Pairs

If $(\alpha_n(a,k;q),\ \beta_n(a,k;q))$ is a WP-Bailey pair, then so are the pairs $(\alpha_n^{'}(a,k;q)$, $\beta_n^{'}(a,k;q))$ given by

$$\alpha_n^{'}(a,k;q) = \frac{(a,\rho_1,\rho_2,-q\sqrt{a};q)_n}{(q,-\sqrt{a},aq/\rho_1,aq/\rho_2;q)_n}\left(\frac{k}{m\sqrt{a}}\right)^n. \tag{3.2.1}$$

$$\beta_n^{'}(a,k;q) = \frac{(mq/\rho_1,mq/\rho_2,k/m,k;q)_n}{(q,aq/\rho_1,aq/\rho_2,mq;q)_n}$$
$$\times\sum_{r=0}^{n}\frac{(q\sqrt{m},-q\sqrt{m},\rho_1,\rho_2,q^{-n},kq^n,m,m/\sqrt{a};q)_r}{(q,q\sqrt{a},\sqrt{m},-\sqrt{m},mq/\rho_1,mq/\rho_2,mq^{1-n}/k,mq^{n+1};q)_r}\left(\frac{q}{\sqrt{a}}\right)^r, \tag{3.2.2}$$

where $m = k\rho_1\rho_2/aq$.

Proof of (3.2.1) - (3.2.2). Let us choose

$$\alpha_r(a,k;q) = \frac{(a,-q\sqrt{a};q)_r}{(q,-\sqrt{a};q)_r}\left(\frac{1}{\sqrt{a}}\right)^r, \tag{3.2.3}$$

in (3.1.2), we obtain

$$\beta_n(a,k;q) = \frac{(k,k/a;q)_n}{(q,aq;q)_n}\sum_{r=0}^{n}\frac{(kq^n,q^{-n},a,-q\sqrt{a};q)_r}{(aq^{1-n}/k,aq^{n+1},q,-\sqrt{a};q)_r}\left(\frac{q\sqrt{a}}{k}\right)^r,$$

by making the use (3.1.18), we have

$$\beta_n(a,k;q) = \frac{(k,k/\sqrt{a};q)_n}{(q,q\sqrt{a};q)_n}\left(\frac{1}{\sqrt{a}}\right)^n. \tag{3.2.4}$$

We obtain new WP-Bailey pair (3.2.3), (3.2.4). Now using WP-Bailey pair (3.2.3) and (3.2.4) in (3.1.3) and (3.1.4), we get (3.2.1), (3.2.2).

$$\alpha_n^{'}(a,k;q) = \frac{(m;q)_{2n}(a,-q\sqrt{a};q)_n}{(k;q)_{2n}(q,-\sqrt{a};q)_n}\left(\frac{k}{m\sqrt{a}}\right)^n. \tag{3.2.5}$$

$$\beta_n^{'}(a,k;q) = \frac{(k/m;q)_n}{(q;q)_n}\sum_{r=0}^{n}\frac{(m,q^{-n},m/\sqrt{a};q)_r}{(q,q\sqrt{a},mq^{1-n}/k;q)_r}\left(\frac{q}{\sqrt{a}}\right)^r, \tag{3.2.6}$$

where $m = qa^2/k$.

Proof of (3.2.5) - (3.2.6). Using WP-Bailey pair (3.2.3), (3.2.4) in (3.1.5) and (3.1.6), we obtain (3.2.5) and (3.2.6).

$$\alpha_n^{'}(a,k;q) = \frac{(qa^2/k;q)_{2n}}{(k;q)_{2n}}\frac{(a,-q\sqrt{a};q)_n}{(q,-\sqrt{a};q)_n}\left(\frac{k^2}{qa^{5/2}}\right)^n. \tag{3.2.7}$$

$$\beta_n^{'}(a,k;q) = \frac{(k^2/qa^2;q)_n}{(q;q)_n}\sum_{r=0}^{n}\frac{(q^{-n},qa^2/k,a^{3/2}q/k;q)_r}{(q,q\sqrt{a},a^2q^{2-n}/k^2;q)_r}\left(\frac{q}{\sqrt{a}}\right)^r. \tag{3.2.8}$$

Proof of (3.2.7) - (3.2.8). By using WP-Bailey pair (3.2.3), (3.2.4) in (3.1.7) and (3.1.8), we get (3.2.7) and (3.2.8).

$$\alpha_n^{'}(a,k;q) = \frac{(\sigma\sqrt{k},-\sigma q\sqrt{m},a,-q\sqrt{a};q)_n(m;q)_{2n}}{(q,-\sqrt{a},\sigma q\sqrt{k},-\sigma\sqrt{m};q)_n(k;q)_{2n}}\left(\frac{k}{m\sqrt{a}}\right)^n. \tag{3.2.9}$$

$$\beta_n^{'}(a,k;q) = \frac{(k/m,\sigma\sqrt{k};q)_n}{(q,q\sigma\sqrt{k};q)_n}\sum_{r=0}^{n}\frac{(m,q^{-n},m/\sqrt{a},-q\sigma\sqrt{m};q)_r}{(q,q\sqrt{a},mq^{1-n}/k,-\sigma\sqrt{m};q)_r}\left(\frac{q}{\sqrt{a}}\right)^r, \tag{3.2.10}$$

where $m = a^2/k, \sigma \in (-1,1)$.

Proof of (3.2.9) - (3.2.10). By making the use of WP-Bailey pair (3.2.3), (3.2.4) in (3.1.9) and (3.1.10), we have (3.2.9), (3.2.10).

$$\alpha_n^{'}(a^2,k;q^2) = \frac{(a,-q\sqrt{a};q)_n}{(q,-\sqrt{a};q)_n}\left(\frac{1}{\sqrt{a}}\right)^n. \tag{3.2.11}$$

$$\beta_n'(a^2,k;q^2) = \frac{(-mq;q)_{2n}(k,k/m^2;q^2)_n}{(-aq;q)_{2n}(q^2,m^2q^2;q^2)_n}\left(\frac{m}{a}\right)^n$$

$$\times\sum_{r=0}^{n}\frac{(m,m/\sqrt{a};q)_r(q^{-2n},kq^{2n},mq^2;q^2)_r}{(q,q\sqrt{a};q)_r(m,m^2q^{2+2n},m^2q^{2-2n}/k;q^2)_r}\left(\frac{mq^2\sqrt{a}}{k}\right)^r, \tag{3.2.12}$$

where $m = k/aq$.

Proof of (3.2.11) - (3.2.12). Now using WP-Bailey pair (3.2.3), (3.2.4) in (3.1.11) and (3.1.12), we get (3.2.11) and (3.2.12).

$$\alpha_n'(a^2,k;q^2) = \frac{(1+aq^{2n})(a,-q\sqrt{a};q)_n}{(1+a)(q,-\sqrt{a};q)_n}\left(\frac{1}{q\sqrt{a}}\right)^n. \tag{3.2.13}$$

$$\beta_n'(a^2,k;q^2) = \frac{(-mq;q)_{2n}(k,k/m^2;q^2)_n}{(-a;q)_{2n}(q^2,m^2q^2;q^2)_n}\left(\frac{m}{aq}\right)^n$$

$$\times\sum_{r=0}^{n}\frac{(m,m/\sqrt{a};q)_r(q^{-2n},kq^{2n},mq^2;q^2)_r}{(q,q\sqrt{a};q)_r(m^2q^{2+2n},m^2q^{2-2n}/k,m;q^2)_r}\left(\frac{mq^2\sqrt{a}}{k}\right)^r, \tag{3.2.14}$$

where $m = k/a$.

Proof of (3.2.13) - (3.2.14). Using WP-Bailey pair (3.2.3), (3.2.4) in (3.1.13) and (3.1.14), we obtain (3.2.13) and (3.2.14).

$$\alpha_{2n}'(a,k;q) = \frac{(a,-q^2\sqrt{a};q^2)_n}{(-\sqrt{a},q^2;q^2)_n}\left(\frac{1}{\sqrt{a}}\right)^n; \quad \alpha_{2n+1}'(a,k;q) = 0. \tag{3.2.15}$$

$$\beta_n'(a,k;q) = \frac{(\sqrt{q}m,-\sqrt{q}m,k,k/m;q)_n}{(\sqrt{q}a,-\sqrt{q}a,q,mq;q)_n}\left(\frac{-k}{a}\right)^n$$

$$\times\sum_{r=0}^{[n/2]}\frac{(1-mq^{4r})(q^{-n},q^{-n+1},kq^n,kq^{n+1},m,m/\sqrt{a};q^2)_r}{(1-m)(mq^{1-n}/k,mq^{2-n}/k,mq^{n+1},mq^{n+2},q^2,q^2\sqrt{a};q^2)_r}\left(\frac{m^2a^{3/2}q^2}{k^4}\right)^r, \tag{3.2.16}$$

where $m = k^2/a$.

Proof of (3.2.15) - (3.2.16). By using WP-Bailey pair (3.2.3), (3.2.4) in (3.1.15) and (3.1.16), we get (3.2.15), (3.2.16).

$$\alpha_n^{'}(a,k;q) = \frac{(\rho_1, \rho_2, a, a\sqrt{q}/m; q)_n}{(q, m\sqrt{q}, aq/\rho_1, aq/\rho_2; q)_n}\left(\frac{kq}{a}\right)^n. \tag{3.2.17}$$

$$\beta_n^{'}(a,k;q) = \frac{(1+\sqrt{a})(k, k/m, mq/\rho_1, mq/\rho_2; q)_n}{(2\sqrt{a})(q, mq, aq/\rho_1, aq/\rho_2; q)_n}$$

$$\times\sum_{r=0}^{n}\frac{(q\sqrt{m}, -q\sqrt{m}, \rho_1, \rho_2, q^{-n}, kq^n, m, \sqrt{q}, m/\sqrt{a}, m\sqrt{q}/\sqrt{a}; q)_r \quad q^r}{(\sqrt{m}, -\sqrt{m}, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{n+1}, q, m\sqrt{q}, \sqrt{aq}, q\sqrt{a}; q)_r}$$

$$-\frac{(1-\sqrt{a})(k, k/m, mq/\rho_1, mq/\rho_2; q)_n}{(2\sqrt{a})(q, mq, aq/\rho_1, aq/\rho_2; q)_n}\sum_{r=0}^{n}\frac{(q\sqrt{m}, -q\sqrt{m}, \rho_1, \rho_2, q^{-n}, kq^n,}{(\sqrt{m}, -\sqrt{m}, mq/\rho_1, mq/\rho_2,}$$

$$\times\frac{m, m/a, \sqrt{q}, m/\sqrt{a}, -m\sqrt{q}/\sqrt{a}; q)_r q^r}{mq^{1-n}/k, mq^{n+1}, q, m\sqrt{q}, -m/a, \sqrt{aq}, -q\sqrt{a}; q)_r} \tag{3.2.18}$$

where $m = k\rho_1\rho_2/aq$.

Proof of (3.2.17) - (3.2.18). Let us choose

$$\alpha_r = \frac{(a, a\sqrt{q}/k; q)_r}{(q, k\sqrt{q}; q)_r}\left(\frac{kq}{a}\right)^r. \tag{3.2.19}$$

in (3.1.2) and by using (3.1.20), we obtain

$$\beta_n(a,k;q) = \frac{(1+\sqrt{a})}{2\sqrt{a}}\frac{(k, \sqrt{q}, k/\sqrt{a}, k\sqrt{q}/\sqrt{a}; q)_n}{(q, k\sqrt{q}, \sqrt{aq}, q\sqrt{a}; q)_n}$$

$$-\frac{(1-\sqrt{a})}{2\sqrt{a}}\frac{(k, k/a, \sqrt{q}, k/\sqrt{a}, -k\sqrt{q}/\sqrt{a}; q)_n}{(q, k\sqrt{q}, -k/a, \sqrt{aq}, -q\sqrt{a}; q)_n}. \tag{3.2.20}$$

We obtain new WP-Bailey pair (3.2.19) and (3.2.20), using (3.2.19), (3.2.20) in (3.1.3) and (3.1.4), we get (3.2.17) and (3.2.18).

$$\alpha_n^{'}(a,k;q)=\frac{(a,a\sqrt{q}/m;q)_n(m;q)_{2n}}{(q,m\sqrt{q};q)_n(k;q)_{2n}}\left(\frac{kq}{a}\right)^n. \tag{3.2.21}$$

$$\beta_n^{'}(a,k;q)=\frac{(1+\sqrt{a})(k/m;q)_n}{(2\sqrt{a})(q;q)_n}\sum_{r=0}^{n}\frac{(m,\sqrt{q},m/\sqrt{a},m\sqrt{q}/\sqrt{a},q^{-n};q)_r\quad q^r}{(q,m\sqrt{q},\sqrt{aq},q\sqrt{a},mq^{1-n}/k;q)_r}$$

$$-\frac{(1-\sqrt{a})(k/m;q)_n}{(2\sqrt{a})(q;q)_n}\sum_{r=0}^{n}\frac{(m,m/a,\sqrt{q},m/\sqrt{a},-m\sqrt{q}/\sqrt{a},q^{-n};q)_r\quad q^r}{(q,m\sqrt{q},-m/a,\sqrt{aq},-q\sqrt{a},mq^{1-n}/k;q)_r}, \tag{3.2.22}$$

where $m=qa^2/k$.

Proof of (3.2.21) - (3.2.22). By making the use (3.2.19), (3.2.20) in (3.1.5) and (3.1.6), we obtain (3.2.21), (3.2.22).

$$\alpha_n^{'}(a,k;q)=\frac{(qa^2/k;q)_{2n}}{(k;q)_{2n}}\frac{(a,k/a\sqrt{q};q)_n}{(q,a^2q^{3/2}/k;q)_n}\left(\frac{kq}{a}\right)^n. \tag{3.2.23}$$

$$\beta_n^{'}(a,k;q)=\frac{(1+\sqrt{a})(k^2/qa^2;q)_n}{(2\sqrt{a})(q;q)_n}\sum_{r=0}^{n}\frac{(q^{-n},\sqrt{q},qa^2/k,a^{3/2}q/k,a^{3/2}q^{3/2}/k;q)_rq^r}{(q,q\sqrt{a},a^2q^{3/2}/k,\sqrt{aq},a^2q^{2-n}/k^2;q)_r}$$

$$-\frac{(1-\sqrt{a})(k^2/qa^2;q)_n}{(2\sqrt{a})(q;q)_n}\sum_{r=0}^{n}\frac{(q^{-n},\sqrt{q},aq/k,qa^2/k,a^{3/2}q/k,-a^{3/2}q^{3/2}/k;q)_rq^r}{(q,-q\sqrt{a},-aq/k,a^2q^{3/2}/k,\sqrt{aq},a^2q^{2-n}/k^2;q)_r}. \tag{3.2.24}$$

Proof of (3.2.23) - (3.2.24). By using (3.2.19), (3.2.20) in (3.1.7) and (3.1.8), we deduce (3.2.23) and (3.2.24).

$$\alpha_n^{'}(a,k;q)=\frac{(1-\sigma\sqrt{k})(1+\sigma\sqrt{mq^n})(a,a\sqrt{q}/m;q)_n(m;q)_{2n}}{(1-\sigma q^n\sqrt{k})(1+\sigma\sqrt{m})(q,m\sqrt{q};q)_n(k;q)_{2n}}\left(\frac{kq}{a}\right)^n. \tag{3.2.25}$$

$$\beta_n^{'}(a,k;q)=\frac{(1+\sqrt{a})(\sigma\sqrt{k},k/m;q)_n}{(2\sqrt{a})(q,\sigma q\sqrt{k};q)_n}$$

$$\times\sum_{r=0}^{n}\frac{(m,\sqrt{q},m/\sqrt{a},m\sqrt{q}/\sqrt{a},-\sigma q\sqrt{m},q^{-n};q)_r q^r}{(q,m\sqrt{q},\sqrt{aq},q\sqrt{a},-\sigma\sqrt{m},mq^{1-n}/k;q)_r}$$

$$-\frac{(1-\sqrt{a})(\sigma\sqrt{k},k/m;q)_n}{(2\sqrt{a})(q,\sigma q\sqrt{k};q)_n}$$

$$\times\sum_{r=0}^{n}\frac{(m,m/a,\sqrt{q},m/\sqrt{a},-m\sqrt{q}/\sqrt{a},-\sigma q\sqrt{m},q^{-n};q)_r \quad q^r}{(q,m\sqrt{q},-m/a,\sqrt{aq},-q\sqrt{a},-\sigma\sqrt{m},mq^{1-n}/k;q)_r}, \tag{3.2.26}$$

where $m=a^2/k, \sigma\in(-1,1)$.

Proof of (3.2.25) - (3.2.26). By making the use (3.2.19), (3.2.20) in (3.1.9) and (3.1.10), we obtain (3.2.25) and (3.2.26).

$$\alpha_n^{'}(a^2,k;q^2)=\frac{(a,a\sqrt{q}/m;q)_n}{(q,m\sqrt{q};q)_n}\left(\frac{mq}{a}\right)^n. \tag{3.2.27}$$

$$\beta_n^{'}(a^2,k;q^2)=\frac{(1+\sqrt{a})(-mq,-mq^2,k/m^2,k;q^2)_n}{(2\sqrt{a})(-aq,-aq^2,q^2,m^2q^2;q^2)_n}\left(\frac{m}{a}\right)^n$$

$$\sum_{r=0}^{n}\frac{(m,\sqrt{q},m/\sqrt{a},m\sqrt{q}/\sqrt{a},q\sqrt{m},-q\sqrt{m},q^{-n},-q^{-n},q^n\sqrt{k},}{(q,m\sqrt{q},\sqrt{aq},q\sqrt{a},\sqrt{m},-\sqrt{m},mq^{1-n}/\sqrt{k},-mq^{1-n}/\sqrt{k},mq^{1+n},}$$

$$\times\frac{-q^n\sqrt{k};q)_r}{-mq^{1+n};q)_r}\left(\frac{amq^2}{k}\right)^r$$

$$-\frac{(1-\sqrt{a})(-mq,-mq^2,k/m^2,k;q^2)_n}{(2\sqrt{a})(-aq,-aq^2,q^2,m^2q^2;q^2)_n}\left(\frac{m}{a}\right)^n$$

$$\sum_{r=0}^{n}\frac{(m,m/a,\sqrt{q},m/\sqrt{a},-m\sqrt{q}/\sqrt{a},q\sqrt{m},-q\sqrt{m},q^{-n},-q^{-n}}{(q,m\sqrt{q},-m/a,\sqrt{aq},-q\sqrt{a},\sqrt{m},-\sqrt{m},mq^{1-n}/\sqrt{k},-mq^{1-n}/\sqrt{k}}$$

$$\times\frac{q^n\sqrt{k},-q^n\sqrt{k};q)_r}{mq^{1+n},-mq^{1+n};q)_r}\left(\frac{amq^2}{k}\right)^r, \tag{3.2.28}$$

where $m=k/aq$.

Proof of (3.2.27) - (3.2.28). Using (3.2.19), (3.2.20) in (3.1.11) and (3.1.12), we get (3.2.27), (3.2.28).

$$\alpha_n^{'}(a^2,k;q^2) = \frac{(a, a\sqrt{q}/m, iq\sqrt{a}, -iq\sqrt{a}; q)_n}{(q, m\sqrt{q}, i\sqrt{a}, -i\sqrt{a}; q)_n}\left(\frac{m}{a}\right)^n. \tag{3.2.29}$$

$$\beta_n^{'}(a^2,k;q^2) = \frac{(1+\sqrt{a})}{(2\sqrt{a})}\frac{(-mq, -mq^2, k/m^2, k; q^2)_n}{(-a, -aq, q^2, m^2q^2; q^2)_n}\left(\frac{m}{aq}\right)^n$$
$$\times\sum_{r=0}^{n}\frac{(m, \sqrt{q}, m/\sqrt{a}, m\sqrt{q}/\sqrt{a}, q\sqrt{m}, -q\sqrt{m}, q^{-n},}{(q, m\sqrt{q}, \sqrt{aq}, q\sqrt{a}, \sqrt{m}, -\sqrt{m}, mq^{1-n}/\sqrt{k},}$$
$$\frac{-q^{-n}, q^n\sqrt{k}, -q^n\sqrt{k}; q)_r}{-mq^{1-n}/\sqrt{k}, mq^{1+n}, -mq^{1+n}; q)_r}\left(\frac{amq^2}{k}\right)^r$$
$$-\frac{(1-\sqrt{a})}{(2\sqrt{a})}\frac{(-mq, -mq^2, k/m^2, k; q^2)_n}{(-a, -aq, q^2, m^2q^2; q^2)_n}\left(\frac{m}{aq}\right)^n$$
$$\times\sum_{r=0}^{n}\frac{(m, m/a, \sqrt{q}, m/\sqrt{a}, -m\sqrt{q}/\sqrt{a}, q\sqrt{m}, -q\sqrt{m}, q^{-n}, -q^{-n},}{(q, -m/a, m\sqrt{q}, \sqrt{aq}, -q\sqrt{a}, \sqrt{m}, -\sqrt{m}, mq^{1-n}/\sqrt{k}, -mq^{1-n}/\sqrt{k},}$$
$$\frac{q^n\sqrt{k}, -q^n\sqrt{k}; q)_r}{mq^{1+n}, -mq^{1+n}; q)_r}\left(\frac{amq^2}{k}\right)^r, \tag{3.2.30}$$

where $m = k/a$.

Proof of (3.2.29) - (3.2.30). By using (3.2.19), (3.2.20) in (3.1.13) and (3.1.14), we obtain (3.2.29) and (3.2.30).

$$\alpha_{2n}^{'}(a,k;q) = \frac{(a, aq/m; q^2)_n}{(q^2, mq; q^2)_n}\left(\frac{mq^2}{a}\right)^n; \qquad \alpha_{2n+1}^{'}(a,k;q) = 0. \tag{3.2.31}$$

$$\beta_n^{'}(a,k;q) = \frac{(1+\sqrt{a})}{(2\sqrt{a})}\frac{(mq;q^2)_n}{(aq;q^2)_n}\frac{(k, k/m; q)_n}{(q, mq; q)_n}\left(\frac{-k}{a}\right)^n$$
$$\times\sum_{r=0}^{[n/2]}\frac{(1-mq^{4r})(q^{-n}, q^{-n+1}, kq^n, kq^{n+1}, m, q, m/\sqrt{a}, mq/\sqrt{a}; q^2)_r}{(1-m)(mq^{1-n}/k, mq^{2-n}/k, mq^{n+1}, mq^{n+2}, q^2, mq, q\sqrt{a}, q^2\sqrt{a}; q^2)_r}\left(\frac{amq}{k^2}\right)^{2r}$$

$$-\frac{(1-\sqrt{a})}{(2\sqrt{a})}\frac{(mq;q^2)_n}{(aq;q^2)_n}\frac{(k,k/m;q)_n}{(q,mq;q)_n}\left(\frac{-k}{a}\right)^n$$

$$\times\sum_{r=0}^{[n/2]}\frac{(1-mq^{4r})(q^{-n},q^{-n+1},kq^n,kq^{n+1},m,m/a,q,m/\sqrt{a},}{(1-m)(mq^{1-n}/k,mq^{2-n}/k,mq^{n+1},mq^{n+2},q^2,mq,-m/a,}$$

$$\times\frac{-mq/\sqrt{a};q^2)_r}{q\sqrt{a},-q^2\sqrt{a};q^2)_r}\left(\frac{amq}{k^2}\right)^{2r}, \tag{3.2.32}$$

where $m = k^2/a$.

Proof of (3.2.31) - (3.2.32). By making the use (3.2.19), (3.2.20) in (3.1.15) and (3.1.16), we get (3.2.31) and (3.2.32).

3.3 Applications

As an application of the new WP-Bailey pairs established in the previous section, we establish a number of basic hypergeometric series identities in this section. If we use the WP-Bailey pairs (3.2.1)-(3.2.32) in (3.1.2), after some simplification we obtain the following presumably new transformations.

$$ {}_8\varphi_7\left[\begin{matrix} q\sqrt{m}, & -q\sqrt{m}, & \rho_1, & \rho_2, & q^{-n}, & kq^n, & m, \\ & \sqrt{m}, & -\sqrt{m}, & mq/\rho_1, & mq/\rho_2, & mq^{1-n}/k, & mq^{n+1}, \end{matrix}\right.$$

$$\left.\begin{matrix} m/\sqrt{a}; & q, & q/\sqrt{a} \\ q\sqrt{a} & & \end{matrix}\right]$$

$$=\frac{(k/a,mq,aq/\rho_1,aq/\rho_2;q)_n}{(aq,mq/\rho_1,mq/\rho_2,k/m;q)_n}$$

$$\times {}_6\varphi_5\left[\begin{matrix} a, & -q\sqrt{a}, & \rho_1, & \rho_2, & kq^n, & q^{-n}; & q, & q\sqrt{a}/m \\ & -\sqrt{a}, & aq/\rho_1, & aq/\rho_2, & aq^{1-n}/k, & aq^{n+1} & & \end{matrix}\right], \tag{3.3.1}$$

where $m = k\rho_1\rho_2/aq$, $|q/\sqrt{a}| < 1$ and $|q\sqrt{a}/m| < 1$.

$$\begin{aligned} &{}_8\varphi_7\left[\begin{matrix} a, & -q\sqrt{a}, & \sqrt{m}, & -\sqrt{m}, & \sqrt{mq}, & -\sqrt{mq}, & kq^n, \\ & -\sqrt{a}, & \sqrt{k}, & -\sqrt{k}, & \sqrt{kq}, & -\sqrt{kq}, & aq^{1-n}/k, \end{matrix}\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad \left.\begin{matrix} q^{-n}; & q, & q\sqrt{a}/m \\ aq^{1+n} & & \end{matrix}\right] \\ &= \frac{(aq, k/m; q)_n}{(k/a, k; q)_n}\ {}_3\varphi_2\left[\begin{matrix} m, & m/\sqrt{a}, & q^{-n}; & q, & q/\sqrt{a} \\ & q\sqrt{a}, & mq^{1-n}/k & & \end{matrix}\right], \end{aligned} \tag{3.3.2}$$

where $m = qa^2/k$, $|q/\sqrt{a}| < 1$ and $|q\sqrt{a}/m| < 1$.

$$\begin{aligned} &{}_8\varphi_7\left[\begin{matrix} a, & -q\sqrt{a}, & q^{-n}, & kq^n, & a\sqrt{q/k}, & -a\sqrt{q/k}, & aq/\sqrt{k}, \\ & -\sqrt{a}, & aq^{n+1}, & aq^{1-n}/k, & \sqrt{k}, & -\sqrt{k}, & \sqrt{kq}, \end{matrix}\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad \left.\begin{matrix} -aq/\sqrt{k}; & q, & k/a^{3/2} \\ -\sqrt{kq} & & \end{matrix}\right] \\ &= \frac{(aq, k^2/qa^2; q)_n}{(k/a, k; q)_n}\ {}_3\Phi_2\left[\begin{matrix} qa^2/k, & qa^{3/2}/k, & q^{-n}; & q, & q/\sqrt{a} \\ & q\sqrt{a}, & a^2q^{2-n}/k^2 & & \end{matrix}\right]. \end{aligned} \tag{3.3.3}$$

where $|q/\sqrt{a}| < 1$ and $|k/a^{3/2}| < 1$.

$$
{}_{10}\varphi_9\left[\begin{matrix} a, & -q\sqrt{a}, & \sqrt{m}, & -\sqrt{m}, & \sqrt{mq}, & -\sqrt{mq}, & \sigma\sqrt{k}, & -q\sigma\sqrt{m}, \\ & -\sqrt{a}, & \sqrt{k}, & -\sqrt{k}, & \sqrt{kq}, & -\sqrt{kq}, & q\sigma\sqrt{k}, & -\sigma\sqrt{m}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} kq^n, & q^{-n}; & q, & q\sqrt{a}/m \\ aq^{1-n}/k, & aq^{n+1} & & \end{matrix}\right]
$$

$$
= \frac{(aq, k/m, \sigma\sqrt{k}; q)_n}{(k/a, k, \sigma q\sqrt{k}; q)_n} \quad {}_4\varphi_3\left[\begin{matrix} m, & m/\sqrt{a}, & q^{-n}, & -\sigma q\sqrt{m}; & q, & q/\sqrt{a} \\ & q\sqrt{a}, & mq^{1-n}/k, & -\sigma\sqrt{m} & & \end{matrix}\right], \tag{3.3.4}
$$

where $m = a^2/k,\ \sigma \in (-1,1),\ |q/\sqrt{a}| < 1$ and $|q\sqrt{a}/m| < 1.$

$$
{}_8\varphi_7\left[\begin{matrix} q\sqrt{m}, & -q\sqrt{m}, & q^{-n}, & -q^{-n}, & q^n\sqrt{k}, & -q^n\sqrt{k}, \\ & \sqrt{m}, & -\sqrt{m}, & mq^{1-n}/\sqrt{k}, & -mq^{1-n}/\sqrt{k}, & mq^{1+n}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} m, & m/\sqrt{a}; & q, & mq^2\sqrt{a}/k \\ -mq^{1+n}, & q\sqrt{a} & & \end{matrix}\right]
$$

$$
= \frac{(m^2q^2, -aq, -aq^2, k, k/a^2; q^2)_n}{(k, -mq, -mq^2, k/m^2, a^2q^2; q^2)_n}\left(\frac{a}{m}\right)^n
$$

$$
\times {}_6\Phi_5\left[\begin{matrix} a, & -q\sqrt{a}, & q^{-n}, & -q^{-n}, & q^n\sqrt{k}, \\ & -\sqrt{a}, & aq^{1-n}/\sqrt{k}, & -aq^{1-n}/\sqrt{k}, & aq^{n+1}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} -q^n\sqrt{k}; & q, & q^2a^{3/2}/k \\ -aq^{n+1} & & \end{matrix}\right], \tag{3.3.5}
$$

where $m = k/aq$, $|mq^2\sqrt{a}/k| < 1$ and $|q^2a^{3/2}/k| < 1$.

$$
{}_8\varphi_7\left[\begin{matrix} m, & m/\sqrt{a}, & q\sqrt{m}, & -q\sqrt{m}, & q^{-n}, & -q^{-n}, \\ & q\sqrt{a}, & \sqrt{m}, & -\sqrt{m}, & mq^{1-n}/\sqrt{k}, & -mq^{1-n}/\sqrt{k}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} q^n\sqrt{k}, & -q^n\sqrt{k}; & q, & mq^2\sqrt{a}/k \\ mq^{1+n}, & -mq^{1+n} & & \end{matrix}\right]
$$

$$
= \frac{(m^2q^2, -aq, -a, k/a^2; q^2)_n}{(-mq, -mq^2, k/m^2, a^2q^2; q^2)_n}\left(\frac{aq}{m}\right)^n
$$

$$
\times {}_6\varphi_5\left[\begin{matrix} iq\sqrt{a}, & -iq\sqrt{a}, & a, & -q\sqrt{a}, & q^{-n}, & -q^{-n}, \\ & i\sqrt{a}, & -i\sqrt{a}, & -\sqrt{a}, & aq^{1-n}/\sqrt{k}, & -aq^{1-n}/\sqrt{k}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} q^n\sqrt{k}, & -q^n\sqrt{k}; & q, & a^{3/2}q/k \\ aq^{n+1}, & -aq^{n+1} & & \end{matrix}\right], \tag{3.3.6}
$$

where $m = k/a$, $|mq^2\sqrt{a}/k| < 1$ and $|a^{3/2}q/k| < 1$

$$
\frac{(1+\sqrt{a})}{2\sqrt{a}} \quad {}_{10}\varphi_9\left[\begin{matrix} q\sqrt{m}, & -q\sqrt{m}, & \rho_1, & \rho_2, & q^{-n}, & kq^n, \\ & \sqrt{m}, & -\sqrt{m}, & mq/\rho_1, & mq/\rho_2, & mq^{1-n}/k, \end{matrix}\right.
$$

$$
\left.\begin{matrix} \sqrt{q}, & m, & m/\sqrt{a}, & m\sqrt{q}/\sqrt{a}; & q, & q \\ mq^{n+1}, & m\sqrt{q}, & \sqrt{aq}, & q\sqrt{a} & & \end{matrix}\right]
$$

$$
-\frac{(1-\sqrt{a})}{2\sqrt{a}} \quad {}_{11}\varphi_{10}\left[\begin{matrix} q\sqrt{m}, & -q\sqrt{m}, & \rho_1, & \rho_2, & q^{-n}, & kq^n, \\ & \sqrt{m}, & -\sqrt{m}, & mq/\rho_1, & mq/\rho_2, & mq^{1-n}/k, \end{matrix}\right.
$$

$$\left.\begin{matrix} \sqrt{q}, & m, & m/a, & m/\sqrt{a}, & -m\sqrt{q}/\sqrt{a}; & q, & q \\ mq^{n+1}, & m\sqrt{q}, & \sqrt{aq}, & -m/a, & -q\sqrt{a} & & \end{matrix}\right]$$

$$= \frac{(aq/\rho_1, aq/\rho_2, mq, k/a; q)_n}{(mq/\rho_1, mq/\rho_2, k/m, aq; q)_n}$$

$$\times {}_6\varphi_5\left[\begin{matrix} \rho_1, & \rho_2, & a, & a\sqrt{q}/m, & q^{-n}, & kq^n; & q, & q^2 \\ & aq/\rho_1, & aq/\rho_2, & m\sqrt{q}, & aq^{1-n}/k, & aq^{n+1} & & \end{matrix}\right], \quad (3.3.7)$$

where $m = k\rho_1\rho_2/aq$.

$$\frac{(1+\sqrt{a})}{2\sqrt{a}} \quad {}_5\varphi_4\left[\begin{matrix} m, & \sqrt{q}, & q^{-n}, & m/\sqrt{a}, & m\sqrt{q}/\sqrt{a}; & q, & q \\ m\sqrt{q}, & \sqrt{aq}, & q\sqrt{a}, & mq^{1-n}/k & & & \end{matrix}\right]$$

$$-\frac{(1-\sqrt{a})}{2\sqrt{a}} \quad {}_6\varphi_5\left[\begin{matrix} m, & \sqrt{q}, & m/a, & q^{-n}, & m/\sqrt{a}, \\ m\sqrt{q}, & -m/a, & \sqrt{aq}, & -q\sqrt{a}, & mq^{1-n}/k \end{matrix}\right.$$

$$\left.\begin{matrix} -m\sqrt{q}/\sqrt{a}; & q, & q \\ & & \end{matrix}\right]$$

$$= \frac{(k, k/a; q)_n}{(aq, k/m; q)_n}$$

$$\times {}_8\varphi_7\left[\begin{matrix} \sqrt{m}, & -\sqrt{m}, & \sqrt{mq}, & -\sqrt{mq}, & a, & a\sqrt{q}/m, \\ & \sqrt{k}, & -\sqrt{k}, & \sqrt{kq}, & -\sqrt{kq}, & m\sqrt{q}, \end{matrix}\right.$$

$$\left.\begin{matrix} kq^n, & q^{-n}; & q, & q^2 \\ aq^{1+n}, & aq^{1-n}/k & & \end{matrix}\right], \quad (3.3.8)$$

where $m = a^2q/k$.

$$
{}_8\varphi_7\left[\begin{matrix} a, & q^{-n}, & kq^n, & k/a\sqrt{q}, & a\sqrt{q/k}, & -a\sqrt{q/k}, \\ & aq^{1-n}/k, & aq^{1+n}, & \sqrt{k}, & -\sqrt{k}, & \sqrt{kq}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} aq/\sqrt{k}, & -aq/\sqrt{k}; & q, & q^2 \\ -\sqrt{kq}, & a^2q^{3/2}/k & & \end{matrix}\right]
$$

$$
= \frac{(1+\sqrt{a})(aq, k^2/qa^2; q)_n}{(2\sqrt{a})(k, k/a; q)_n}
$$

$$
\times {}_5\varphi_4\left[\begin{matrix} \sqrt{q}, & qa^2/k, & qa^{3/2}/k, & q^{3/2}a^{3/2}/k, & q^{-n} & ; q & , q \\ & \sqrt{aq}, & q\sqrt{a}, & a^2q^{2-n}/k^2, & a^2q^{3/2}/k & & \end{matrix}\right]
$$

$$
-\frac{(1-\sqrt{a})(aq, k^2/qa^2; q)_n}{(2\sqrt{a})(k, k/a; q)_n}
$$

$$
\times {}_6\varphi_5\left[\begin{matrix} \sqrt{q}, & qa^2/k, & aq/k, & -q^{3/2}a^{3/2}/k, & qa^{3/2}/k, \\ & \sqrt{aq}, & -q\sqrt{a}, & a^2q^{2-n}/k^2, & a^2q^{3/2}/k, \end{matrix}\right.
$$

$$
\left.\begin{matrix} q^{-n}; & q, & q \\ -aq/k & & \end{matrix}\right]. \quad (3.3.9)
$$

$$
\frac{(1+\sqrt{a})}{2\sqrt{a}} \quad {}_6\varphi_5\left[\begin{matrix} -\sigma q\sqrt{m}, & q^{-n}, & m, & \sqrt{q}, & m/\sqrt{a}, \\ & -\sigma\sqrt{m}, & m\sqrt{q}, & \sqrt{aq}, & q\sqrt{a}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} m\sqrt{q}/\sqrt{a}; & q, & q \\ mq^{1-n}/k & & \end{matrix}\right]
$$

$$
-\frac{(1-\sqrt{a})}{2\sqrt{a}} \quad {}_7\varphi_6\left[\begin{matrix} -\sigma q\sqrt{m}, & m, & m/a, & \sqrt{q}, & m/\sqrt{a}, & q^{-n}, \\ & -\sigma\sqrt{m}, & m\sqrt{q}, & -m/a, & \sqrt{aq}, & -q\sqrt{a}, \end{matrix}\right.
$$

$$\left.\begin{matrix} -m\sqrt{q}/\sqrt{a}; \quad q, \quad q \\ mq^{1-n}/k \end{matrix}\right]$$

$$= \frac{(\sigma q\sqrt{k}, k/a, k; q)_n}{(\sigma\sqrt{k}, k/m, aq; q)_n}$$

$$\times {}_{10}\varphi_9\left[\begin{matrix} \sigma\sqrt{k}, & -\sigma q\sqrt{m}, & \sqrt{m}, & -\sqrt{m}, & \sqrt{mq}, & -\sqrt{mq}, \\ & \sigma q\sqrt{k}, & -\sigma\sqrt{m}, & \sqrt{k}, & -\sqrt{k}, & \sqrt{kq}, \end{matrix}\right.$$

$$\left.\begin{matrix} a, & a\sqrt{q/m}, & q^{-n}, & kq^n; & q, & q^2 \\ -\sqrt{kq}, & m\sqrt{q}, & aq^{1-n}/k, & aq^{n+1} & & \end{matrix}\right], \tag{3.3.10}$$

where $m = a^2/k, \ \sigma \in (-1, 1)$.

$$\frac{(1+\sqrt{a})}{2\sqrt{a}} \quad {}_{10}\varphi_9\left[\begin{matrix} m, & q\sqrt{m}, & -q\sqrt{m}, & m/\sqrt{a}, & m\sqrt{q}/\sqrt{a}, \\ & \sqrt{m}, & -\sqrt{m}, & mq^{1-n}/\sqrt{k}, & -mq^{1-n}/\sqrt{k}, \end{matrix}\right.$$

$$\left.\begin{matrix} q^{-n}, & -q^n, & \sqrt{q}, & q^n\sqrt{k}, & -q^n\sqrt{k}; & q, & amq^2/k \\ mq^{n+1}, & -mq^{n+1}, & m\sqrt{q}, & \sqrt{aq}, & q\sqrt{a} & & \end{matrix}\right]$$

$$-\frac{(1-\sqrt{a})}{2\sqrt{a}} \ {}_{11}\varphi_{10}\left[\begin{matrix} m, & q\sqrt{m}, & -q\sqrt{m}, & \sqrt{q}, & m/a, & -m\sqrt{q}/\sqrt{a}, \\ & \sqrt{m}, & -\sqrt{m}, & -m/a, & m\sqrt{q}, & \sqrt{aq}, \end{matrix}\right.$$

$$\left.\begin{matrix} m/\sqrt{a} & q^{-n}, & -q^n, & q^n\sqrt{k}, & -q^n\sqrt{k}; & q, & amq^2/k \\ -q\sqrt{a}, & mq^{n+1}, & -mq^{n+1}, & mq^{1-n}/\sqrt{k}, & -mq^{1-n}/\sqrt{k} & & \end{matrix}\right]$$

$$= \frac{(-aq, -aq^2, m^2q^2, k/a^2; q^2)_n}{(-mq, -mq^2, k/m^2, a^2q^2; q^2)_n}\left(\frac{a}{m}\right)^n$$

$$\times {}_6\varphi_5\left[\begin{matrix} a, & a\sqrt{q}/m, & q^{-n}, & -q^{-n}, & q^n\sqrt{k}, \\ & m\sqrt{q}, & aq^{1-n}/\sqrt{k}, & -aq^{1-n}/\sqrt{k}, & aq^{n+1} \end{matrix}\right.$$

$$\left.\begin{matrix} -q^n\sqrt{k}; & q, & amq^3/k \\ -aq^{n+1} & & \end{matrix}\right], \tag{3.3.11}$$

where $m = k/aq$, $|amq^3/k| < 1$ and $|amq^2/k| < 1$.

$$\frac{(1+\sqrt{a})}{2\sqrt{a}} \quad {}_{10}\varphi_9\left[\begin{matrix} m, & q\sqrt{m}, & -q\sqrt{m}, & \sqrt{q}, & m/\sqrt{a}, & m\sqrt{q}/\sqrt{a}, \\ & \sqrt{m}, & -\sqrt{m}, & m\sqrt{q}, & \sqrt{aq}, & q\sqrt{a}, \end{matrix}\right.$$

$$\left.\begin{matrix} q^n\sqrt{k}, & -q^n\sqrt{k}, & q^{-n}, & -q^n; & q, & amq^2/k \\ mq^{1-n}/\sqrt{k}, & -mq^{1-n}/\sqrt{k}, & mq^{n+1}, & -mq^{n+1} & & \end{matrix}\right]$$

$$-\frac{(1-\sqrt{a})}{2\sqrt{a}} \quad {}_{11}\varphi_{10}\left[\begin{matrix} m, & q\sqrt{m}, & -q\sqrt{m}, & \sqrt{q}, & m/a, & m/\sqrt{a}, \\ & \sqrt{m}, & -\sqrt{m}, & -m/a, & m\sqrt{q}, & \sqrt{aq}, \end{matrix}\right.$$

$$\left.\begin{matrix} q^{-n}, & -q^n, & -m\sqrt{q}/\sqrt{a}, & q^n\sqrt{k}, & -q^n\sqrt{k}; & q, & amq^2/k \\ -q\sqrt{a}, & mq^{1-n}/\sqrt{k}, & -mq^{1-n}/\sqrt{k}, & mq^{n+1}, & -mq^{n+1} & & \end{matrix}\right]$$

$$= \frac{(-a, -aq, q^2, m^2q^2; q^2)_n (k, k/a; q)_n}{(-mq, -mq^2, k/m^2, k; q^2)_n (q, aq; q)_n} \left(\frac{aq}{m}\right)^n$$

$$\times {}_8\varphi_7\left[\begin{matrix} a, & a\sqrt{q}/m, & iq\sqrt{a}, & -iq\sqrt{a}, & q^{-n}, & -q^{-n}, \\ & i\sqrt{a}, & -i\sqrt{a}, & m\sqrt{q}, & aq^{1-n}/\sqrt{k}, & -aq^{1-n}/\sqrt{k}, \end{matrix}\right.$$

$$\left.\begin{matrix} q^n\sqrt{k}, & -q^n\sqrt{k}; & q, & amq^2/k \\ aq^{n+1}, & -aq^{n+1} & & \end{matrix}\right], \tag{3.3.12}$$

where $m = k/a$ and $|amq^2/k| < 1$.

Chapter 4

Transformations and Summations of Basic Hypergeometric Series

4.1 Introduction

In the previous two chapters we have used the concept of Bailey lemma and pair to develop the summation and transformation identities of basic hypergeometric series. The development of the present chapter is based based on and elegant generalization of the concept of Bailey lemma by Andrews [17]. In particular, in the present chapter we have established five identities connecting the components α_n and β_n of WP-Bailey pair (α_n, β_n). We then have used the known Bailey pairs to derive some summations and transformations identities of basic hypergeometric series.

We recall from chapter 1 that a WP-Bailey pair is the pair of sequence satis-

The content of this chapter is based on the reference [9].

fying

$$\beta_n(a,k;q)=\sum_{r=0}^{n}\frac{(k/a;q)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}}\ \alpha_r(a,k;q). \tag{4.1.1}$$

This identity follows by setting

$$u_r=\frac{(k/a;q)_r}{(q;q)_r}; \qquad\qquad v_r=\frac{(k;q)_r}{(aq;q)_r}$$

in (1.4.1). Also the same substitution in (1.4.10) gives

$$\gamma_n=\frac{(k;q)_{2n}}{(aq;q)_{2n}}\sum_{r=0}^{\infty}\frac{(k/a;q)_r(kq^{2n};q)_r}{(q;q)_r(aq^{2n+1};q)_r}\delta_{r+n}. \tag{4.1.2}$$

It may be noted that for $k=0$ in (4.1.1), we get the standard Bailey pair (1.4.9).

In the next sections, we shall required the definition of following WP-Bailey pairs.

We define a WP-Bailey unit Bailey pair as

$$\alpha_n=\frac{(a,q\sqrt{a},-q\sqrt{a},a/k;q)_n}{(q,\sqrt{a},-\sqrt{a},kq;q)_n}\left(\frac{k}{a}\right)^n, \tag{4.1.3}$$

$$\beta_n=\begin{cases}1, & n=0,\\ 0, & n>0.\end{cases}$$

The trivial WP-Bailey pair is defined as

$$\beta_n=\frac{(k,k/a;q)_n}{(q,aq;q)_n}, \tag{4.1.4}$$

$$\alpha_n = \begin{cases} 1, & n = 0. \\ 0, & n > 0, \end{cases}$$

A WP-Bailey pair due to Singh [80] is

$$\alpha_n = \frac{(a, q\sqrt{a}, -q\sqrt{a}, y, z, a^2q/kyz; q)_n}{(q, \sqrt{a}, -\sqrt{a}, aq/y, aq/z, kyz/a; q)_n} \left(\frac{k}{a}\right)^n,$$

$$\beta_n = \frac{(ky/a, kz/a, k, aq/yz; q)_n}{(q, aq/y, aq/z, kyz/a; q)_n}. \tag{4.1.5}$$

In our analysis we shall also require the following known results

$${}_4\varphi_3\left[\begin{matrix} a, & -q\sqrt{a}, & b, & c; & q, & q\sqrt{a}/bc \\ & -\sqrt{a}, & aq/b, & aq/c & & \end{matrix}\right]$$

$$= \frac{(aq, q\sqrt{a}/b, q\sqrt{a}/c, aq/bc; q)_\infty}{(aq/b, aq/c, q\sqrt{a}, q\sqrt{a}/bc; q)_\infty}. \tag{4.1.6}$$

$([44], II.13)$

$${}_3\varphi_2\left[\begin{matrix} a, & \lambda q, & b; & q, & \lambda^2/ab^2 \\ & \lambda, & q\lambda^2/b & & \end{matrix}\right]$$

$$= \frac{1 - \lambda + \lambda/b(1 - \lambda/a)}{(1 - \lambda)(1 + \lambda/b)} \frac{(\lambda^2/b^2, q\lambda^2/ab; q)_\infty}{(q\lambda^2/b, \lambda^2/ab^2; q)_\infty}, \tag{4.1.7}$$

where $|\lambda^2/ab^2| < 1$.

[44]

$$
{}_2\varphi_1\left[\begin{matrix} a, & b; & q, & -q/b \\ & aq/b & & \end{matrix}\right] = \frac{(-q;q)_\infty (aq, aq^2/b^2; q^2)_\infty}{(-q/b, aq/b; q)_\infty}, \tag{4.1.8}
$$

where $|q/b| < 1$.

([44] II.9)

$$
{}_4\varphi_3\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b; & q, & 1/b^2q \\ & \sqrt{a}, & -\sqrt{a}, & aq/b & & \end{matrix}\right] = \frac{(a/b^2, 1/bq; q)_\infty}{(aq/b, 1/b^2q; q)_\infty}. \tag{4.1.9}
$$

[96]

$$
{}_8\varphi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & \sqrt{a/b}, & -\sqrt{a/b}, & \sqrt{aq/b}, & -\sqrt{aq/b}, & b; & q, & bq \\ & \sqrt{a}, & -\sqrt{a}, & q\sqrt{ab}, & -q\sqrt{ab}, & \sqrt{abq}, & -\sqrt{abq}, & aq/b & & \end{matrix}\right]
$$

$$
= \frac{(aq, b^2q; q)_\infty}{(bq, abq; q)_\infty}, \tag{4.1.10}
$$

where $|bq| < 1$.

[62]

4.2 Main Results

If (α_n, β_n) is a WP-Bailey pair, then under suitable convergence conditions the following relations are true

$$
\sum_{n=0}^{\infty} \frac{(-q\sqrt{k}, c; q)_n}{(-\sqrt{k}, kq/c; q)_n} \left(\frac{aq}{c\sqrt{k}}\right)^n \beta_n
$$

$$= \frac{(kq, aq/\sqrt{k}, q\sqrt{k}/c, aq/c; q)_\infty}{(aq/c\sqrt{k}, aq, kq/c, q\sqrt{k}; q)_\infty} \sum_{n=0}^{\infty} \frac{(k;q)_{2n}(-q\sqrt{k}, q\sqrt{k}, c; q)_n}{(kq;q)_{2n}(-\sqrt{k}, aq/\sqrt{k}, aq/c; q)_n} \left(\frac{aq}{c\sqrt{k}}\right)^n \alpha_n. \tag{4.2.1}$$

$$\sum_{n=0}^{\infty} \frac{(q^{n+1}\sqrt{ak};q)_n}{(q^n\sqrt{ak};q)_n} \left(\frac{a^2}{k^2}\right)^n \beta_n$$

$$= \frac{(a/k, a^2q/k; q)_\infty}{(a^2/k^2, aq; q)_\infty} \sum_{n=0}^{\infty} \frac{(k, kq^n, q^{n+1}\sqrt{ak}; q)_n}{(q^n\sqrt{ak}, a^2q/k, a^2q^{n+1}/k; q)_n}$$

$$\times \frac{(1-\sqrt{ak}q^{2n}+\sqrt{a}/(\sqrt{k}-a^{3/2}q^{2n}))}{(1-q^{2n}\sqrt{ak})(1+\sqrt{a}/\sqrt{k})} \left(\frac{a^2}{k^2}\right)^n \alpha_n. \tag{4.2.2}$$

$$\sum_{n=0}^{\infty} \frac{(q\sqrt{k}, -q\sqrt{k}; q)_n}{(\sqrt{k}, -\sqrt{k}; q)_n} \left(\frac{a^2}{k^2 q}\right)^n \beta_n$$

$$= \frac{(a^2/k, a/kq; q)_\infty}{(aq, a^2/k^2q; q)_\infty} \sum_{n=0}^{\infty} \frac{(k, kq^n, q\sqrt{k}, -q\sqrt{k}; q)_n}{(a^2/k, a^2q^n/k, \sqrt{k}, -\sqrt{k}; q)_n} \left(\frac{a^2}{qk^2}\right)^n \alpha_n. \tag{4.2.3}$$

$$\sum_{n=0}^{\infty} \frac{(q\sqrt{k}, -q\sqrt{k}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q)_n}{(\sqrt{k}, -\sqrt{k}, -kq\sqrt{1/a}, kq\sqrt{1/a}, k\sqrt{q/a}, -k\sqrt{q/a}; q)_n} \left(\frac{kq}{a}\right)^n \beta_n$$

$$= \frac{(kq, k^2q/a^2; q)_\infty}{(kq/a, k^2q/a; q)_\infty} \sum_{n=0}^{\infty} \frac{(k, kq^n, q\sqrt{k}, -q\sqrt{k}, \sqrt{a}, -\sqrt{a}, \sqrt{aq},}{(\sqrt{k}, -\sqrt{k}, kq\sqrt{1/a}, -kq\sqrt{1/a}, k\sqrt{q/a},}$$

$$\frac{-\sqrt{aq}, k^2q/a, k^2q^{n+1}/a; q)_n}{-k\sqrt{q/a}, aq, aq^{n+1}, kq, kq^{n+1}; q)_n} \left(\frac{kq}{a}\right)^n \alpha_n. \tag{4.2.4}$$

$$\sum_{n=0}^{\infty}\left(\frac{-aq}{k}\right)^n\beta_n$$

$$=\frac{(kq,a^2q^2/k;q^2)_\infty(-q;q)_\infty}{(-aq/k,aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(k,kq^n;q)_n}{(kq,a^2q^2/k;q^2)_{2n}}\left(\frac{-aq}{k}\right)^n\alpha_n. \quad (4.2.5)$$

Proof of (4.2.1). Substituting $a = kq^{2n}, b = k/a$ and $c = cq^n$ in (4.1.6), we have

$${}_4\varphi_3\left[\begin{array}{cccccc} kq^{2n}, & -q^{n+1}\sqrt{k}, & k/a, & cq^n; & q, & aq/c\sqrt{k} \\ & -q^n\sqrt{k}, & aq^{2n+1}, & kq^{n+1}/c & & \end{array}\right]$$

$$=\frac{(kq,aq/\sqrt{k},q\sqrt{k}/c,aq/c;q)_\infty(aq;q)_{2n}(kq/c,q\sqrt{k};q)_n}{(aq/c\sqrt{k},aq,kq/c,q\sqrt{k};q)_\infty(kq;q)_{2n}(aq/\sqrt{k},aq/c;q)_n}. \quad (4.2.6)$$

Putting

$$\delta_r=\frac{(c,-q\sqrt{k};q)_r}{(-\sqrt{k},kq/c;q)_r}\left(\frac{aq}{c\sqrt{k}}\right)^r$$

in (4.1.2) and making the use of (4.2.6), we get

$$\gamma_n=\frac{(k;q)_{2n}(q\sqrt{k},-q\sqrt{k},c;q)_n(kq,aq/\sqrt{k},q\sqrt{k}/c,aq/c;q)_\infty}{(kq;q)_{2n}(-\sqrt{k},aq/\sqrt{k},aq/c;q)_n(aq,aq/c\sqrt{k},kq/c,q\sqrt{k};q)_\infty}\left(\frac{aq}{c\sqrt{k}}\right)^n.$$

Substituting δ_n and γ_n as above in (1.4.3), we get (4.2.1).

Proof of (4.2.2). Setting $a = k/a, b = kq^{2n}$ and $\lambda = q^{2n}\sqrt{ak}$ in (4.1.7), we get

$${}_3\varphi_2\left[\begin{array}{ccccc} k/a, & q^{2n+1}\sqrt{ak}, & kq^{2n}; & q, & a^2/k^2 \\ & q^{2n}\sqrt{ak}, & aq^{2n+1} & & \end{array}\right]$$

$$=\frac{(a/k,a^2q/k;q)_\infty(aq;q)_{2n}}{(a^2/k^2,aq;q)_\infty(a^2q/k;q)_{2n}}\frac{(1-q^{2n}\sqrt{ak}+\sqrt{a}/(\sqrt{k}-a^{3/2}q^{2n}))}{(1-q^{2n}\sqrt{ak})(1+\sqrt{a}/\sqrt{k})}, \quad (4.2.7)$$

where $|a^2/k^2| < 1$.

Choosing

$$\delta_r = \frac{(q^{n+1}\sqrt{ak};q)_r}{(q^n\sqrt{ak};q)_r}\left(\frac{a^2}{k^2}\right)^r$$

in (4.1.2) and with the use of (4.2.7), we have

$$\gamma_n = \frac{(1 - q^{2n}\sqrt{ak} + \sqrt{a}/(\sqrt{k} - a^{3/2}q^{2n}))}{(1 - q^{2n}\sqrt{ak})(1 + \sqrt{a}/\sqrt{k})}$$

$$\times \frac{(a/k, a^2q/k;q)_\infty(k, kq^n, q^{n+1}\sqrt{ak};q)_n}{(aq, a^2/k^2;q)_\infty(q^n\sqrt{ak}, a^2q/k, a^2q^{n+1}/k;q)_n}\left(\frac{a^2}{k^2}\right)^n.$$

Using δ_n and γ_n in (1.4.3), we obtain (4.2.2).

Proof of (4.2.3). Choosing $a = kq^{2n}$ and $b = k/a$ in (4.1.9), we get

$${}_4\varphi_3\left[\begin{matrix} kq^{2n}, & q^{n+1}\sqrt{k}, & -q^{n+1}\sqrt{k}, & k/a; & q, & a^2/k^2q \\ & q^n\sqrt{k}, & -q^n\sqrt{k}, & aq^{2n+1} & & \end{matrix}\right]$$

$$= \frac{(a^2q^{2n}/k, a/kq;q)_\infty}{(aq^{2n+1}, a^2/k^2q;q)_\infty}. \quad (4.2.8)$$

Now taking

$$\delta_r = \frac{(q\sqrt{k}, -q\sqrt{k};q)_r}{(\sqrt{k}, -\sqrt{k};q)_r}\left(\frac{a^2}{k^2q}\right)^r$$

in (4.1.2) and using (4.2.8), we get

$$\gamma_n = \frac{(a^2/k, a/kq;q)_\infty(k;q)_{2n}(q\sqrt{k}, -q\sqrt{k};q)_n}{(aq, a^2/qk^2;q)_\infty(a^2/k;q)_{2n}(\sqrt{k}, -\sqrt{k};q)_n}\left(\frac{a^2}{qk^2}\right)^n.$$

substituting δ_n and γ_n in (1.4.3), we obtain (4.2.3).

Proof of (4.2.4). Taking $a = kq^{2n}$ and $b = k/a$ in (4.1.10), we get

$$
{}_8\varphi_7\left[\begin{matrix} kq^{2n}, & q^{n+1}\sqrt{k}, & -q^{n+1}\sqrt{k}, & q^n\sqrt{a}, & -q^n\sqrt{a}, \\ & q^n\sqrt{k}, & -q^n\sqrt{k}, & kq^{n+1}/\sqrt{a}, & -kq^{n+1}/\sqrt{a}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} q^n\sqrt{aq}, & -q^n\sqrt{aq}, & k/a; & q, & kq/a \\ kq^n\sqrt{q/a}, & -kq^n\sqrt{q/a}, & aq^{2n+1} & & \end{matrix}\right]
$$

$$
= \frac{(kq^{2n+1}, k^2q/a^2; q)_\infty}{(kq/a, k^2q^{2n+1}/a; q)_\infty}, \quad (4.2.9)
$$

where $|kq/a| < 1$.

Putting

$$
\delta_r = \frac{(q\sqrt{k}, -q\sqrt{k}, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q)_r}{(\sqrt{k}, -\sqrt{k}, kq\sqrt{1/a}, -kq\sqrt{1/a}, k\sqrt{q/a}, -k\sqrt{q/a}; q)_r}\left(\frac{kq}{a}\right)^r
$$

in (4.1.2) and applying (4.2.9), we have

$$
\gamma_n = \frac{(kq, qk^2/a^2; q)_\infty (qk^2/a, k; q)_{2n} (q\sqrt{k}, -q\sqrt{k}, \sqrt{a}, -\sqrt{a}, \sqrt{aq},}{(kq/a, k^2q/a; q)_\infty (kq, aq; q)_{2n} (\sqrt{k}, -\sqrt{k}, kq\sqrt{1/a}, -kq\sqrt{1/a},}
$$

$$
\times \frac{-\sqrt{aq}; q)_n}{k\sqrt{q/a}, -k\sqrt{q/a}; q)_n}\left(\frac{kq}{a}\right)^n.
$$

Substituting δ_n and γ_n in (1.4.3), we get (4.2.4).

Proof of (4.2.5). Setting $a = kq^{2n}$ and $b = k/a$ in (4.1.8), we get

$$
{}_2\varphi_1\left[\begin{matrix} kq^{2n}, & k/a & ; & q, & -aq/k \\ & aq^{2n+1} & & & \end{matrix}\right]
$$

$$= \frac{(-q;q)_\infty (kq^{2n+1}, a^2q^{2n+2}/k; q^2)_\infty}{(-aq/k, aq^{2n+1}; q)_\infty}. \quad (4.2.10)$$

Taking

$$\delta_r = \left(\frac{-aq}{k}\right)^r$$

in (4.1.2) and using (4.2.10), we get

$$\gamma_n = \frac{(-q;q)_\infty (kq, a^2q^2/k; q^2)_\infty (k, kq^n; q)_n}{(aq, -aq/k; q)_\infty (kq, a^2q^2/k; q^2)_{2n}} \left(\frac{-aq}{k}\right)^n.$$

Substituting δ_n and γ_n in (1.4.3), we get (4.2.5).

4.3 Applications

By using (4.1.3) in (4.2.1) and taking $n \to \infty$, we get

$${}_8\varphi_7\left[\begin{matrix} k, & q\sqrt{k}, & -q\sqrt{k}, & c, & a, & q\sqrt{a}, & -q\sqrt{a}, & a/k; & q, & q\sqrt{k}/c \\ & kq, & -\sqrt{k}, & aq/\sqrt{k}, & \sqrt{a}, & -\sqrt{a}, & kq, & aq/c & & \end{matrix}\right]$$

$$= \frac{(q\sqrt{k}, kq/c, aq, aq/c\sqrt{k}; q)_\infty}{(kq, aq/\sqrt{k}, q\sqrt{k}/c, aq/c; q)_\infty}. \quad (4.3.1)$$

where $|q\sqrt{k}/c| < 1$

Again by making the use of (4.1.5) in (4.2.1) and taking $n \to \infty$, we obtain

$${}_{10}\varphi_9\left[\begin{matrix} k, & -q\sqrt{k}, & c, & q\sqrt{k}, & a, & q\sqrt{a}, & -q\sqrt{a}, & y, & z, \\ & kq, & -\sqrt{k}, & aq/\sqrt{k}, & aq/c, & \sqrt{a}, & -\sqrt{a}, & aq/y, & aq/z, \end{matrix}\right.$$

$$\left.\begin{matrix} a^2q/kyz; & q, & q\sqrt{k}/c \\ kyz/a & & \end{matrix}\right]$$

$$= \frac{(aq/c\sqrt{k}, aq, kq/c, q\sqrt{k}; q)_\infty}{(kq, aq/\sqrt{k}, q\sqrt{k}/c, aq/c; q)_\infty}$$

$$\times {}_6\varphi_5\left[\begin{matrix} -q\sqrt{k}, & c, & ky/a, & kz/a, & k, & aq/yz; & q, & aq/c\sqrt{k} \\ & -\sqrt{k}, & kq/c, & aq/y, & aq/z, & kyz/a & & \end{matrix}\right], \tag{4.3.2}$$

where $|q\sqrt{k}/c| < 1$ and $|aq/c\sqrt{k}| < 1$.

On using (4.1.4) in (4.2.2) and taking $n \to \infty$, we get

$${}_2\varphi_1\left[\begin{matrix} k, & k/a; & q, & a^2/k^2 \\ & aq & & \end{matrix}\right]$$

$$= \frac{(a/k, a^2q/k; q)_\infty}{(aq, a^2/k^2; q)_\infty}\frac{(1-\sqrt{ak}+\sqrt{a}/(\sqrt{k}-a^{3/2})}{(1-\sqrt{ak})(1+\sqrt{a}/\sqrt{k})}, \tag{4.3.3}$$

where $|a^2/k^2| < 1$.

By making the use of (4.1.3) in (4.2.3) and then taking $n \to \infty$, we obtain

$${}_7\varphi_6\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & a/k, & k, & q\sqrt{k}, & -q\sqrt{k}; & q, & a/kq \\ & \sqrt{a}, & -\sqrt{a}, & kq, & a^2/k, & \sqrt{k}, & -\sqrt{k} & & \end{matrix}\right]$$

$$= \frac{(aq, a^2/k^2q; q)_\infty}{(a^2/k, a/kq; q)_\infty}, \tag{4.3.4}$$

where $|a/kq| < 1$.

In (4.2.3) using (4.1.5) and then taking $n \to \infty$, we get the following transformation

$$
{}_{9}\varphi_{8}\left[\begin{array}{cccccccccc} k, & q\sqrt{k}, & -q\sqrt{k}, & a, & q\sqrt{a}, & -q\sqrt{a}, & y, & z, & a^2q/kyz; & q, \; a/kq \\ & a^2/k, & \sqrt{k}, & -\sqrt{k}, & \sqrt{a}, & -\sqrt{a}, & aq/y, & aq/z, & kyz/a & \end{array}\right]
$$

$$
= \frac{(aq, a^2/k^2q; q)_\infty}{(a^2/k, a/kq; q)_\infty}
$$

$$
{}_{6}\varphi_{5}\left[\begin{array}{ccccccc} q\sqrt{k}, & -q\sqrt{k}, & ky/a, & kz/a, & k, & aq/yz; & q, \; a^2/k^2q \\ & \sqrt{k}, & -\sqrt{k}, & aq/y, & aq/z, & kyz/a & \end{array}\right], \tag{4.3.5}
$$

where $|a/kq| < 1$ and $|a^2/k^2q| < 1$.

Again in (4.2.4) making the use of (4.1.3) and taking $n \to \infty$, we obtain the following summation

$$
{}_{12}\varphi_{11}\left[\begin{array}{cccccccccccc} k, & q\sqrt{k}, & -q\sqrt{k}, & \sqrt{a}, & -\sqrt{a}, & \sqrt{aq}, & -\sqrt{aq}, & k^2q/a, & a, & q\sqrt{a}, & -q\sqrt{a}, & a/k; \; q, \; k^2q/a^2 \\ & \sqrt{k}, & -\sqrt{k}, & kq\sqrt{1/a}, & -kq\sqrt{1/a}, & k\sqrt{q/a}, & -k\sqrt{q/a}, & \sqrt{a}, & -\sqrt{a}, & kq, & aq, & kq \end{array}\right]
$$

$$
= \frac{(kq/a, k^2q/a; q)_\infty}{(kq, k^2q/a^2; q)_\infty} \tag{4.3.6}
$$

Now use (4.1.5) in (4.2.4) and taking $n \to \infty$,we have

$$
{}_{14}\varphi_{13}\left[\begin{array}{ccccccc} k, & q\sqrt{k}, & -q\sqrt{k}, & \sqrt{a}, & -\sqrt{a}, & \sqrt{aq}, & -\sqrt{aq}, \\ & \sqrt{k}, & -\sqrt{k}, & kq\sqrt{1/a}, & -kq\sqrt{1/a}, & k\sqrt{q/a}, & -k\sqrt{q/a}, \end{array}\right.
$$

$$\left.\begin{matrix} k^2q/a, & a, & q\sqrt{a}, & -q\sqrt{a}, & y, & z, & a^2q/kyz; & q, & k^2q/a^2 \\ aq, & kq & \sqrt{a}, & -\sqrt{a}, & aq/y, & aq/z, & kyz/a & & \end{matrix}\right]$$

$$= \frac{(kq/a, k^2q/a; q)_\infty}{(kq, k^2q/a^2; q)_\infty}$$

$$\times {}_{10}\varphi_9\left[\begin{matrix} q\sqrt{k}, & -q\sqrt{k}, & \sqrt{a}, & -\sqrt{a}, & \sqrt{aq}, & -\sqrt{aq}, & ky/a, \\ & \sqrt{k}, & -\sqrt{k}, & kq\sqrt{1/a}, & -kq\sqrt{1/a}, & k\sqrt{q/a}, & -k\sqrt{q/a}, \end{matrix}\right.$$

$$\left.\begin{matrix} kz/a, & k, & aq/yz; & q, & kq/a \\ aq/y, & aq/z, & kyz/a & & \end{matrix}\right], \tag{4.3.7}$$

where max $(|k^2q/a^2|, |kq/a|) < 1$.

By using (4.1.3) in (4.2.5), we have

$$\sum_{n=0}^{\infty} \frac{(k, kq^n, a, q\sqrt{a}, -q\sqrt{a}, a/k; q)_n(-q)^n}{(q, \sqrt{a}, -\sqrt{a}, kq; q)_n(kq, a^2q^2/k; q^2)_{2n}}$$

$$= \frac{(-aq/k, aq; q)_\infty}{(-q; q)_\infty(kq, a^2q^2/k; q^2)_\infty}. \tag{4.3.8}$$

and again in (4.2.5) using (4.1.5), we get

$${}_4\varphi_3\left[\begin{matrix} ky/a, & kz/a, & k, & aq/yz; & q, & -aq/k \\ & aq/y, & aq/z, & kyz/a & & \end{matrix}\right] = \frac{(-q; q)_\infty(kq, a^2q^2/k; q^2)_\infty}{(-aq/k, aq; q)_\infty}$$

$$\times \sum_{n=0}^{\infty} \frac{(k, kq^n, a, q\sqrt{a}, -q\sqrt{a}, y, z, a^2q/kyz; q)_n(-q)^n}{(kq, a^2q^2/k; q^2)_{2n}(q, \sqrt{a}, -\sqrt{a}, aq/y, aq/z, kyz/a; q)_n}, \tag{4.3.9}$$

where $|aq/k| < 1$.

Chapter 5

Transformations and Summations of Bilateral Basic Hypergeometric Series

5.1 Introduction

The method of Cauchy [30] in the second proof of Jaccobi triple product identity [51] has been successfully used to prove various transformations of basic bilateral hypergeometric series by many authors. By this method (refer to as Cauchy's method of bilateralization) we can obtain summations and transformations of bilateral basic hypergeometric series starting from a known terminating unilateral basic hypergeometric series. For the earlier use of this method one may refer to the work of Bailey [26], Slater [87] and Shukla [77]. Recently, Cauchy's method and its variants have been used by Jouhet

The content of this chapter is based on the reference [10].

and Schlosser [53, 54], Chen and Fu [32], Schlosser [75], Jouhet [55], Zhang [104], Zhang and Zhang [103], Zhang and Hu [105] and Antony [23] in driving a number of bilateral basic hypergeometric series identities.

In the present chapter, we have applied the Cauchy's method of bilateralization on the following terminating series identities [96] to get the bilateral extension of the same.

$$
{}_{12}\varphi_{11}\left[\begin{matrix} k, & q\sqrt{k}, & -q\sqrt{k}, & kb/a, & kc/a, & kd/a, & -q\sqrt{a}, & \sqrt{a}, \\ & \sqrt{k}, & -\sqrt{k}, & aq/b, & aq/c, & aq/d, & -k/\sqrt{a}, & kq/\sqrt{a}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} \sqrt{aq}, & -\sqrt{aq}, & k^2q^n/a, & q^{-n}; & q, & q \\ k\sqrt{q/a}, & -k\sqrt{q/a} & kq^{1+n}, & aq^{1-n}/k & & \end{matrix}\right]
$$

$$
= \frac{(kq, k^2/a^2, k/\sqrt{a}; q)_n}{(k/a, k^2/a, kq/\sqrt{a}; q)_n}
$$

$$
\times {}_6\varphi_5\left[\begin{matrix} a, & -q\sqrt{a}, & b, & c, & d, & q^{-n}; & q, & q \\ & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & a^2q^{1-n}/k^2 & & \end{matrix}\right], \tag{5.1.1}
$$

where $k = a^2q/bcd$.

$$
{}_{12}\varphi_{11}\left[\begin{matrix} k, & q\sqrt{k}, & -q\sqrt{k}, & kb/a, & kc/a, & kd/a, & q\sqrt{a}, & -\sqrt{a}, \\ & \sqrt{k}, & -\sqrt{k}, & aq/b, & aq/c, & aq/d, & k/\sqrt{a}, & -kq/\sqrt{a}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} \sqrt{aq}, & -\sqrt{aq}, & k^2q^n/a, & q^{-n}; & q, & q \\ k\sqrt{q/a}, & -k\sqrt{q/a} & kq^{1+n}, & aq^{1-n}/k & & \end{matrix}\right]
$$

$$= \frac{(kq, k^2/a^2, -k/\sqrt{a}; q)_n}{(k/a, k^2/a, -kq/\sqrt{a}; q)_n}$$

$$\times {}_6\varphi_5 \left[\begin{matrix} a, & q\sqrt{a}, & b, & c, & d, & q^{-n}; & q, & q \\ & \sqrt{a}, & aq/b, & aq/c, & aq/d, & a^2q^{1-n}/k^2 & & \end{matrix} \right], \tag{5.1.2}$$

where $k = a^2q/bcd$.

$${}_{10}\varphi_9 \left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & a/d, & a^2q^{1+n}/d, & \sqrt{d}, & -\sqrt{d}, \\ & \sqrt{a}, & -\sqrt{a}, & dq, & dq^{-n}/a, & aq/\sqrt{d}, & -aq/\sqrt{d}, \end{matrix} \right.$$

$$\left. \begin{matrix} \sqrt{dq}, & -\sqrt{dq}, & q^{-n}; & q, & q \\ a\sqrt{q/d}, & -a\sqrt{q/d}, & aq^{1+n} & & \end{matrix} \right]$$

$$= \frac{(aq, a^2q/d^2; q)_n}{(aq/d, a^2q/d; q)_n}. \tag{5.1.3}$$

$${}_6\varphi_5 \left[\begin{matrix} a^2/q, & a\sqrt{q}, & -a\sqrt{q}, & a/b\sqrt{q}, & c, & q^{-n}; & q, & q \\ & a/\sqrt{q}, & -a/\sqrt{q}, & ab\sqrt{q}, & e, & a^2cq^{1-n}/b^2e & & \end{matrix} \right]$$

$$= \frac{(e/c, eb^2/a^2; q)_n}{(e, eb^2/ca^2; q)_n}$$

$${}_6\varphi_5 \left[\begin{matrix} b^2/q, & b\sqrt{q}, & -b\sqrt{q}, & b/a\sqrt{q}, & c, & q^{-n}; & q, & q \\ & b/\sqrt{q}, & -b/\sqrt{q}, & ab\sqrt{q}, & eb^2/a^2, & cq^{1-n}/e & & \end{matrix} \right]. \tag{5.1.4}$$

5.2 Main Results

The following transformations are true whenever the series involved are convergent

$$
{}_{12}\psi_{12}\left[\begin{matrix} \sqrt{a}, & \sqrt{aq}, & -\sqrt{aq}, & q\sqrt{k}, & -q\sqrt{k}, & -q\sqrt{a}, & kb/a, & kc/a, \\ \sqrt{k}, & -\sqrt{k}, & aq/b, & aq/c, & -k/\sqrt{a}, & kq/\sqrt{a}, & k\sqrt{q/a}, & -k\sqrt{q/a}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} k^2q^n/a, & kq^{-n}, & q^{-n}, & kdq^n/a; & q, & q \\ q^{1+n}, & kq^{1+n}, & aq^{1-n}/d, & aq^{1-n}/k & & \end{matrix}\right]
$$

$$
= \frac{(q/b, q/c, q/k, k^2/a^2, k/\sqrt{a}, d, kq; q)_n}{(aq/kb, aq/kc, q/a, k/a, kq/\sqrt{a}, kd/a, k^2/a; q)_n}
$$

$$
{}_6\psi_6\left[\begin{matrix} b, & c, & -q\sqrt{a}, & dq^n, & aq^{-n}, & q^{-n}; & q, & q \\ q^{1+n}, & -\sqrt{a} & aq/b, & aq/c, & aq^{1-n}/d, & a^2q^{1-n}/k^2 & & \end{matrix}\right], \quad (5.2.1)
$$

where $k = a^2q/bcd$.

$$
{}_{12}\psi_{12}\left[\begin{matrix} -\sqrt{a}, & \sqrt{aq}, & -\sqrt{aq}, & q\sqrt{k}, & -q\sqrt{k}, & q\sqrt{a}, & kb/a, & kc/a, \\ \sqrt{k}, & -\sqrt{k}, & aq/b, & aq/c, & k/\sqrt{a}, & -kq/\sqrt{a}, & k\sqrt{q/a}, & -k\sqrt{q/a}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} k^2q^n/a, & kq^{-n}, & q^{-n}, & kdq^n/a; & q, & q \\ q^{1+n}, & kq^{1+n}, & aq^{1-n}/d, & aq^{1-n}/k & & \end{matrix}\right]
$$

$$
= \frac{(q/b, q/c, q/k, k^2/a^2, -k/\sqrt{a}, d, kq; q)_n}{(aq/kb, aq/kc, q/a, k/a, -kq/\sqrt{a}, kd/a, k^2/a; q)_n}
$$

$$
{}_6\psi_6\left[\begin{matrix} b, & c, & q\sqrt{a}, & dq^n, & aq^{-n}, & q^{-n}; & q, & q \\ q^{1+n}, & \sqrt{a}, & aq/b, & aq/c, & aq^{1-n}/d, & a^2q^{1-n}/k^2 & & \end{matrix}\right], \quad (5.2.2)
$$

where $k = a^2q/bcd$.

$$
{}_{10}\psi_{10}\left[\begin{matrix} aq^{-n}, & q\sqrt{a}, & -q\sqrt{a}, & aq^n/d, & a^2q^{1+n}/d, & \sqrt{d}, & -\sqrt{d}, \\ q^{1+n}, & \sqrt{a}, & -\sqrt{a}, & dq^{1-n}, & dq^{-n}/a, & aq/\sqrt{d}, & -aq/\sqrt{d}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} \sqrt{dq}, & -\sqrt{dq}, & q^{-n}; & q, & q \\ a\sqrt{q/d}, & -a\sqrt{q/d}, & aq^{1+n} & & \end{matrix}\right]
$$

$$
= \frac{(a\sqrt{q/d}, -a\sqrt{q/d}, -aq/d, aq, 1/\sqrt{d}, -1/\sqrt{d}, q/a, q; q)_n}{(a/d, a^2q/d, q/\sqrt{d}, -q/\sqrt{d}, -q, -\sqrt{q}, \sqrt{q}, 1/d; q)_n}. \tag{5.2.3}
$$

$$
{}_6\psi_6\left[\begin{matrix} a\sqrt{q}, & -a\sqrt{q}, & c, & aq^{n-1/2}/b, & a^2q^{-n-1}, & q^{-n}; & q, & q \\ q^{n+1}, & a/\sqrt{q}, & -a/\sqrt{q}, & e, & abq^{-n+1/2}, & a^2cq^{1-n}/b^2e & & \end{matrix}\right]
$$

$$
= \frac{(eb^2/a^2, b/a\sqrt{q}, q^{1/2}/b, -q^{1/2}/b, e/c, q^2/a^2, q/b, -q/b; q)_n}{(e, a/b\sqrt{q}, q^{1/2}/a, -q^{1/2}/a, b^2e/a^2c, q^2/b^2, q/a, -q/a; q)_n}
$$

$$
{}_6\psi_6\left[\begin{matrix} b\sqrt{q}, & -b\sqrt{q}, & c, & bq^{n-1/2}/a, & b^2q^{-n-1}, & q^{-n}; & q, & q \\ q^{n+1}, & b/\sqrt{q}, & -b/\sqrt{q}, & abq^{-n+1/2}, & eb^2/a^2, & cq^{1-n}/e & & \end{matrix}\right]. \tag{5.2.4}
$$

Proof of (5.2.1). Replacing $n \to 2n$ in (5.1.1), we have

$$
\sum_{r=0}^{2n} \frac{(a, -q\sqrt{a}, b, c, d, q^{-2n}; q)_r q^r}{(q, -\sqrt{a}, aq/b, aq/c, aq/d, a^2q^{1-2n}/k^2; q)_r}
$$

$$
= \frac{(k/a, k^2/a, kq/\sqrt{a}; q)_{2n}}{(kq, k^2/a^2, k/\sqrt{a}; q)_{2n}} \sum_{r=0}^{2n} \frac{(k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, kd/a,}{(q, \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, -k/\sqrt{a},}
$$

$$
\times \frac{-q\sqrt{a}, \sqrt{a}, \sqrt{aq}, -\sqrt{aq}, k^2q^{2n}/a, q^{-2n}; q)_r q^r}{kq/\sqrt{a}, k\sqrt{q/a}, -k\sqrt{q/a}, aq^{1-2n}/k, kq^{1+2n}; q)_r}.
$$

Now taking $r \to s+n$ and we have the following

$$\sum_{s=-n}^{n} \frac{(a, -q\sqrt{a}, b, c, d, q^{-2n}; q)_{s+n} q^{s+n}}{(q, -\sqrt{a}, aq/b, aq/c, aq/d, a^2q^{1-2n}/k^2; q)_{s+n}} = \frac{(k/a, k^2/a, kq/\sqrt{a}; q)_{2n}}{(kq, k^2/a^2, k/\sqrt{a}; q)_{2n}}$$

$$\times \sum_{s=-n}^{n} \frac{(k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, kd/a, -q\sqrt{a}, \sqrt{a},}{(q, \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, -k/\sqrt{a}, kq/\sqrt{a}, k\sqrt{q/a},}$$

$$\times \frac{\sqrt{aq}, -\sqrt{aq}, k^2q^{2n}/a, q^{-2n}; q)_{s+n} q^{s+n}}{-k\sqrt{q/a}, aq^{1-2n}/k, kq^{1+2n}; q)_{s+n}}.$$

Replacing $a \longrightarrow aq^{-2n}$, $k \longrightarrow kq^{-2n}$, $b \longrightarrow bq^{-n}$ and $c \longrightarrow cq^{-n}$, we get desired result (5.2.1).

Proof of (5.2.2). Changing $n \to 2n$ in (5.1.2), we have

$$\sum_{r=0}^{2n} \frac{(a, q\sqrt{a}, b, c, d, q^{-2n}; q)_r q^r}{(q, \sqrt{a}, aq/b, aq/c, aq/d, a^2q^{1-2n}/k^2; q)_r} = \frac{(k/a, k^2/a, -kq/\sqrt{a}; q)_{2n}}{(kq, k^2/a^2, -k/\sqrt{a}; q)_{2n}}$$

$$\times \sum_{r=0}^{2n} \frac{(k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, kd/a, q\sqrt{a}, -\sqrt{a},}{(q, \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, k/\sqrt{a}, -kq/\sqrt{a}, k\sqrt{q/a},}$$

$$\times \frac{\sqrt{aq}, -\sqrt{aq}, k^2q^{2n}/a, q^{-2n}; q)_r q^r}{-k\sqrt{q/a}, aq^{1-2n}/k, kq^{1+2n}; q)_r}.$$

and then replacing $r \to s+n$, we get

$$\sum_{s=-n}^{n} \frac{(a, q\sqrt{a}, b, c, d, q^{-2n}; q)_{s+n} q^{s+n}}{(q, \sqrt{a}, aq/b, aq/c, aq/d, a^2q^{1-2n}/k^2; q)_{s+n}} = \frac{(k/a, k^2/a, -kq/\sqrt{a}; q)_{2n}}{(kq, k^2/a^2, -k/\sqrt{a}; q)_{2n}}$$

$$\times \sum_{s=-n}^{n} \frac{(k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, kd/a, q\sqrt{a}, -\sqrt{a},}{(q, \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, k/\sqrt{a}, -kq/\sqrt{a}, k\sqrt{q/a},}$$

$$\times \frac{\sqrt{aq}, -\sqrt{aq}, k^2q^{2n}/a, q^{-2n}; q)_{s+n} q^{s+n}}{-k\sqrt{q/a}, aq^{1-2n}/k, kq^{1+2n}; q)_{s+n}}.$$

Substituting $a \longrightarrow aq^{-2n}$, $k \longrightarrow kq^{-2n}$, $b \longrightarrow bq^{-n}$ and $c \longrightarrow cq^{-n}$ in above, we obtain (5.2.2).

Proof of (5.2.3). Replacing $n \to 2n$ in (5.1.3), we have

$$\sum_{r=0}^{2n} \frac{(a, q\sqrt{a}, -q\sqrt{a}, a/d, a^2q^{1+2n}/d, \sqrt{d}, -\sqrt{d}, \sqrt{dq}, -\sqrt{dq}, q^{-2n}; q)_r q^r}{(q, \sqrt{a}, -\sqrt{a}, dq, dq^{-2n}/a, aq/\sqrt{d}, -aq/\sqrt{d}, a\sqrt{q/d}, -a\sqrt{q/d}, aq^{1+2n}; q)_r}$$

$$= \frac{(aq, a^2q/d^2; q)_{2n}}{(aq/d, a^2q/d; q)_{2n}}.$$

Replacing $r \to s+n$ then changing $a \to aq^{-2n}$, $d \to dq^{-2n}$ and after some simplification we get (5.2.3).

Proof of (5.2.4). Changing n to $2n$ in (5.1.4)

$$\sum_{r=0}^{2n} \frac{(a^2/q, a\sqrt{q}, -a\sqrt{q}, a/b\sqrt{q}, c, q^{-2n}; q)_r}{(a/\sqrt{q}, -a/\sqrt{q}, ab\sqrt{q}, e, a^2cq^{1-2n}/b^2e; q)_r} q^r = \frac{(e/c, eb^2/a^2; q)_{2n}}{(e, eb^2/ca^2; q)_{2n}} \times \sum_{r=0}^{2n} \frac{(b^2/q, b\sqrt{q}, -b\sqrt{q}, b/a\sqrt{q}, c, q^{-2n}; q)_r}{(b/\sqrt{q}, -b/\sqrt{q}, ab\sqrt{q}, eb^2/a^2, cq^{1-n}/e; q)_r} q^r.$$

and taking $r \to s+n$, we get

$$\sum_{s=-n}^{n} \frac{(a^2/q, a\sqrt{q}, -a\sqrt{q}, a/b\sqrt{q}, c, q^{-2n}; q)_{s+n}}{(a/\sqrt{q}, -a/\sqrt{q}, ab\sqrt{q}, e, a^2cq^{1-2n}/b^2e; q)_{s+n}} q^{s+n} = \frac{(e/c, eb^2/a^2; q)_{2n}}{(e, eb^2/ca^2; q)_{2n}} \times \sum_{s=-n}^{n} \frac{(b^2/q, b\sqrt{q}, -b\sqrt{q}, b/a\sqrt{q}, c, q^{-2n}; q)_{s+n}}{(b/\sqrt{q}, -b/\sqrt{q}, ab\sqrt{q}, eb^2/a^2, cq^{1-n}/e; q)_{s+n}} q^{s+n}.$$

then replacing $a \to aq^{-n}, b \to bq^{-n}, c \to cq^{-n}$ and $e \to eq^{-n}$, we get (5.2.4).

5.3 Some Special Case

In this section, we mention some interesting special cases of our main results.

By putting $d = q$ in (5.2.1),we get

$$
{}_{12}\psi_{12}\left[\begin{matrix} \sqrt{a}, & \sqrt{aq}, & -\sqrt{aq}, & q\sqrt{k}, & -q\sqrt{k}, & -q\sqrt{a}, & kb/a, \\ \sqrt{k}, & -\sqrt{k}, & aq/b, & aq/c, & -k/\sqrt{a}, & kq/\sqrt{a}, & k\sqrt{q/a}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} kc/a, & k^2q^n/a, & kq^{-n}, & q^{-n} & kq^{1+n}/a; & q & q \\ -k\sqrt{q/a}, & q^{1+n}, & kq^{1+n}, & aq^{-n}, & aq^{1-n}/k & & \end{matrix}\right]
$$

$$
= \frac{(q/b, q/c, q/k, k^2/a^2, k/\sqrt{a}, q, kq; q)_n}{(aq/kb, aq/kc, q/a, k/a, kq/\sqrt{a}, kq/a, k^2/a; q)_n}
$$

$$
{}_4\psi_4\left[\begin{matrix} b, & c, & -q\sqrt{a}, & q^{-n}; & q, & q \\ -\sqrt{a}, & aq/b, & aq/c, & a^2q^{1-n}/k^2 & & \end{matrix}\right], \quad (5.3.1)
$$

where $k = a^2/bc$.

Subtituting $b = a$ in (5.3.1), we get

$$
{}_4\varphi_3\left[\begin{matrix} a, & c, & -q\sqrt{a}, & q^{-n}; & q, & q \\ & -\sqrt{a}, & aq/c, & a^2q^{1-n}/k^2 & & \end{matrix}\right]
$$

$$
= \frac{(aq/kc, k/a, kq/\sqrt{a}, kq/a, k^2/a; q)_n}{(q/c, k^2/a^2, k/\sqrt{a}, q, kq; q)_n}, \quad (5.3.2)
$$

where $a = kc$.

Subtituting $d = aq^{-n}$ in (5.2.1), we obtain

$$ {}_{12}\varphi_{11}\left[\begin{matrix} \sqrt{a}, & \sqrt{aq}, & -\sqrt{aq}, & q\sqrt{k}, & -q\sqrt{k}, & -q\sqrt{a}, & kb/a, \\ & \sqrt{k}, & -\sqrt{k}, & aq/b, & aq/c, & -k/\sqrt{a}, & kq/\sqrt{a}, \end{matrix}\right. $$

$$ \left.\begin{matrix} kc/a, & k^2q^n/a, & kq^{-n}, & q^{-n}, & k; & q, & q \\ k\sqrt{q/a}, & -k\sqrt{q/a}, & q^{1+n}, & kq^{1+n}, & aq^{1-n}/k & & \end{matrix}\right] $$

$$ = \frac{(q/b, q/c, q/k, k^2/a^2, k/\sqrt{a}, aq^{-n}, kq; q)_n}{(aq/kb, aq/kc, q/a, k/a, kq/\sqrt{a}, kq^{-n}, k^2/a; q)_n} $$

$$ {}_6\varphi_5\left[\begin{matrix} a, & b, & c, & aq^{-n}, & -q\sqrt{a}, & q^{-n}; & q, & q \\ & q^{1+n}, & -\sqrt{a}, & aq/b, & aq/c, & a^2q^{1-n}/k^2 & & \end{matrix}\right], \quad (5.3.3) $$

where $k = aq^{n+1}/bc$.

By putting $b = a$ in (5.2.1), we get

$$ {}_{12}\varphi_{11}\left[\begin{matrix} \sqrt{a}, & \sqrt{aq}, & -\sqrt{aq}, & q\sqrt{k}, & -q\sqrt{k}, & -q\sqrt{a}, & k, \\ & \sqrt{k}, & -\sqrt{k}, & aq/c, & -k/\sqrt{a}, & kq/\sqrt{a}, & k\sqrt{q/a}, \end{matrix}\right. $$

$$ \left.\begin{matrix} kc/a, & k^2q^n/a, & kq^{-n}, & q^{-n}, & kdq^n/a; & q, & q \\ -k\sqrt{q/a}, & q^{1+n}, & kq^{1+n}, & aq^{1-n}/d, & aq^{1-n}/k & & \end{matrix}\right] $$

$$ = \frac{(q/c, k^2/a^2, k/\sqrt{a}, d, kq; q)_n}{(aq/kc, k/a, kq/\sqrt{a}, kd/a, k^2/a; q)_n} $$

$$ {}_6\varphi_5\left[\begin{matrix} a, & c, & -q\sqrt{a}, & dq^n, & aq^{-n}, & q^{-n}; & q, & q \\ & q^{1+n}, & -\sqrt{a}, & aq/c, & aq^{1-n}/d, & a^2q^{1-n}/k^2 & & \end{matrix}\right], \quad (5.3.4) $$

where $k = aq/cd$.

By taking $d = aq$ in (5.2.1),we obtain

$$
{}_{10}\psi_{10}\left[\begin{matrix} \sqrt{a}, & \sqrt{aq}, & -\sqrt{aq}, & q\sqrt{k}, & -q\sqrt{k}, & -q\sqrt{a}, & kb/a, \\ \sqrt{k}, & -\sqrt{k}, & aq/b, & aq/c, & -k/\sqrt{a}, & kq/\sqrt{a}, & k\sqrt{q/a}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} kc/a, & k^2q^n/a, & kq^{-n}; & q, & q \\ -k\sqrt{q/a}, & q^{1+n}, & aq^{1-n}/k & & \end{matrix}\right]
$$

$$
= \frac{(q/b, q/c, q/k, k^2/a^2, k/\sqrt{a}, aq; q)_n}{(aq/kb, aq/kc, q/a, k/a, kq/\sqrt{a}, k^2/a; q)_n}
$$

$$
{}_{5}\psi_{5}\left[\begin{matrix} b, & c, & -q\sqrt{a}, & aq^{1+n}, & aq^{-n}; & q, & q \\ q^{1+n}, & -\sqrt{a}, & aq/b, & aq/c, & a^2q^{1-n}/k^2 & & \end{matrix}\right], \quad (5.3.5)
$$

where $k = a/bc$.

Setting $b = \sqrt{a}$ and $c = -\sqrt{a}$ in (5.2.1), we obtain

$$
{}_{10}\psi_{10}\left[\begin{matrix} \sqrt{a}, & \sqrt{aq}, & -\sqrt{aq}, & q\sqrt{k}, & -q\sqrt{k}, & k/\sqrt{a}, & k^2q^n/a, \\ \sqrt{k}, & -\sqrt{k}, & q\sqrt{a}, & kq/\sqrt{a}, & k\sqrt{q/a}, & -k\sqrt{q/a}, & q^{1+n}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} kq^{-n}, & q^{-n}, & kdq^n/a; & q, & q \\ kq^{1+n}, & aq^{1-n}/d, & aq^{1-n}/k & & \end{matrix}\right]
$$

$$
= \frac{(q/\sqrt{a}, -q/\sqrt{a}, q/k, k^2/a^2, k/\sqrt{a}, d, kq; q)_n}{(q\sqrt{a}/k, -q\sqrt{a}/k, q/a, k/a, kq/\sqrt{a}, kd/a, k^2/a; q)_n}
$$

$$
{}_4\psi_4\left[\begin{matrix} \sqrt{a}, & dq^n, & aq^{-n}, & q^{-n}; & q, & q \\ q^{1+n}, & q\sqrt{a}, & aq^{1-n}/d, & a^2q^{1-n}/k^2 & & \end{matrix}\right], \quad (5.3.6)
$$

where $k = -aq/d$.

Setting $d = q$ in (5.2.2), we have

$$
{}_{12}\psi_{12}\left[\begin{matrix} kq^{-n}, & q\sqrt{k}, & -q\sqrt{k}, & kb/a, & kc/a, & kq^{1+n}/a, & q\sqrt{a}, & -\sqrt{a}, \\ q^{1+n}, & \sqrt{k}, & -\sqrt{k}, & aq/b, & aq/c, & aq^{-n}, & k/\sqrt{a}, & -kq/\sqrt{a}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} \sqrt{aq}, & -\sqrt{aq}, & k^2q^n/a, & q^{-n}; & q, & q \\ k\sqrt{q/a}, & -k\sqrt{q/a}, & aq^{1-n}/k, & kq^{1+n} & & \end{matrix}\right]
$$

$$
= \frac{(q/b, q/c, q/k, k^2/a^2, -k/\sqrt{a}, q, kq; q)_n}{(aq/kb, aq/kc, q/a, k/a, -kq/\sqrt{a}, kq/a, k^2/a; q)_n}
$$

$$
{}_4\psi_4\left[\begin{matrix} b, & c, & q\sqrt{a}, & q^{-n}; & q, & q \\ \sqrt{a}, & aq/b, & aq/c, & a^2q^{1-n}/k^2 & & \end{matrix}\right], \quad (5.3.7)
$$

where $k = a^2/bc$.

Substituting $b = a$ in (5.3.7), we obtain

$$
{}_4\varphi_3\left[\begin{matrix} a, & c, & q\sqrt{a}, & q^{-n}; & q, & q \\ & \sqrt{a}, & aq/c, & a^2q^{1-n}/k^2 & & \end{matrix}\right]
$$

$$
= \frac{(aq/kc, k/a, -kq/\sqrt{a}, kq/a, k^2/a; q)_n}{(q/c, k^2/a^2, -k/\sqrt{a}, q, kq; q)_n}, \quad (5.3.8)
$$

where $a = kc$.

Taking $aq^{-n} = d$ in (5.2.2), we get

$$
{}_{12}\varphi_{11}\left[\begin{matrix} kq^{-n}, & q\sqrt{k}, & -q\sqrt{k}, & kb/a, & kc/a, & k, & q\sqrt{a}, & -\sqrt{a}, \\ & q^{1+n}, & \sqrt{k}, & -\sqrt{k}, & aq/b, & aq/c, & k/\sqrt{a}, & -kq/\sqrt{a}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} \sqrt{aq}, & -\sqrt{aq}, & k^2q^n/a, & q^{-n}; & q, & q \\ k\sqrt{q/a}, & -k\sqrt{q/a}, & kq^{1+n}, & aq^{1-n}/k & & \end{matrix}\right]
$$

$$
= \frac{(q/b, q/c, q/k, k^2/a^2, -k/\sqrt{a}, aq^{-n}, kq; q)_n}{(aq/kb, aq/kc, q/a, k/a, -kq/\sqrt{a}, kq^{-n}, k^2/a; q)_n}
$$

$$
{}_6\varphi_5\left[\begin{matrix} a, & b, & c, & aq^{-n}, & q\sqrt{a}, & q^{-n}; & q, & q \\ & q^{1+n}, & \sqrt{a}, & aq/b, & aq/c, & a^2q^{1-n}/k^2 & & \end{matrix}\right], \quad (5.3.9)
$$

where $k = aq^{n+1}/bc$.

Substituting $d = aq$ in (5.2.2), we obtain

$$
{}_{10}\psi_{10}\left[\begin{matrix} kq^{-n}, & q\sqrt{k}, & -q\sqrt{k}, & kb/a, & kc/a, & q\sqrt{a}, & -\sqrt{a}, & \sqrt{aq}, \\ q^{1+n}, & \sqrt{k}, & -\sqrt{k}, & aq/b, & aq/c, & k/\sqrt{a}, & -kq/\sqrt{a}, & k\sqrt{(q/a)}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} -\sqrt{aq}, & k^2q^n/a; & q, & q \\ -k\sqrt{(q/a)}, & aq^{1-n}/k & & \end{matrix}\right]
$$

$$
= \frac{(q/b, q/c, q/k, k^2/a^2, -k/\sqrt{a}, aq; q)_n}{(aq/kb, aq/kc, q/a, k/a, -kq/\sqrt{a}, k^2/a; q)_n}
$$

$$
{}_5\psi_5\left[\begin{matrix} aq^{-n}, & b, & c, & q\sqrt{a}, & aq^{1+n}; & q, & q \\ q^{1+n}, & \sqrt{a}, & aq/b, & aq/c, & a^2q^{1-n}/k^2 & & \end{matrix}\right], \quad (5.3.10)
$$

where $k = a/bc$.

By putting $b = a$ in (5.2.2), we get

$$
{}_{12}\varphi_{11}\left[\begin{matrix} -\sqrt{a}, & \sqrt{aq}, & -\sqrt{aq}, & q\sqrt{k}, & -q\sqrt{k}, & q\sqrt{a}, \\ & \sqrt{k}, & -\sqrt{k}, & aq/c, & k/\sqrt{a}, & -kq/\sqrt{a}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} k, & kc/a, & k^2q^n/a, & kq^{-n}, & q^{-n}, & kdq^n/a; & q, & q \\ k\sqrt{q/a}, & -k\sqrt{q/a}, & q^{1+n}, & kq^{1+n}, & aq^{1-n}/d, & aq^{1-n}/k & & \end{matrix}\right]
$$

$$
= \frac{(q/c, k^2/a^2, -k/\sqrt{a}, d, kq; q)_n}{(aq/kc, k/a, -kq/\sqrt{a}, kd/a, k^2/a; q)_n}
$$

$$
{}_6\varphi_5\left[\begin{matrix} a, & c, & q\sqrt{a}, & dq^n, & aq^{-n}, & q^{-n}; & q, & q \\ & q^{1+n}, & \sqrt{a}, & aq/c, & aq^{1-n}/d, & a^2q^{1-n}/k^2 & & \end{matrix}\right], \quad (5.3.11)
$$

where $k = aq/cd$.

Setting $b = \sqrt{a}$ and $c = -\sqrt{a}$ in (5.2.2), we obtain

$$
{}_{10}\psi_{10}\left[\begin{matrix} -\sqrt{a}, & \sqrt{aq}, & -\sqrt{aq}, & q\sqrt{k}, & -q\sqrt{k}, & -k/\sqrt{a}, \\ \sqrt{k}, & -\sqrt{k}, & -q\sqrt{a}, & -kq/\sqrt{a}, & k\sqrt{q/a}, & -k\sqrt{q/a}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} k^2q^n/a, & kq^{-n}, & q^{-n}, & kdq^n/a; & q, & q \\ q^{1+n}, & kq^{1+n}, & aq^{1-n}/d, & aq^{1-n}/k & & \end{matrix}\right]
$$

$$= \frac{(q/\sqrt{a}, -q/\sqrt{a}, q/k, k^2/a^2, -k/\sqrt{a}, d, kq; q)_n}{(q\sqrt{a/k}, -q\sqrt{a/k}, q/a, k/a, -kq/\sqrt{a}, kd/a, k^2/a; q)_n}$$

$$ {}_4\psi_4 \left[\begin{matrix} -\sqrt{a}, & dq^n, & aq^{-n}, & q^{-n}; & q, & q \\ q^{1+n}, & -q\sqrt{a}, & aq^{1-n}/d, & a^2q^{1-n}/k^2 & & \end{matrix} \right], \quad (5.3.12)$$

where $k = -aq/d$.

Putting $d = a^2$ in (5.2.3), we get

$$ {}_8\varphi_7 \left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & q^n/a, & a, & -a, & a\sqrt{q}, & -a\sqrt{q}, & q^{-n}; & q, & q \\ & \sqrt{a}, & -\sqrt{a}, & a^2q^{1-n}, & -q, & \sqrt{q}, & -\sqrt{q}, & aq^{1+n} & & \end{matrix} \right]$$

$$= \frac{(\sqrt{q}/a, -\sqrt{q}/a, aq, -1/a; q)_n}{(-q, \sqrt{q}, -\sqrt{q}, 1/a^2; q)_n}. \quad (5.3.13)$$

Choosing $e = q$ in (5.2.4), we get

$$ {}_6\psi_6 \left[\begin{matrix} b\sqrt{q}, & -b\sqrt{q}, & c, & bq^{n-1/2}/a, & b^2q^{-n-1}, & q^{-n}; & q, & q \\ q^{n+1}, & b/\sqrt{q}, & -b/\sqrt{q}, & abq^{-n+1/2}, & b^2q/a^2, & cq^{-n} & & \end{matrix} \right]$$

$$= \frac{(q, a/bq^{1/2}, q^{1/2}/a, -q^{1/2}/a, b^2q/a^2c, q^2/b^2, q/a, -q/a; q)_n}{(b^2q/a^2, b/aq^{1/2}, q^{1/2}/b, -q^{1/2}/b, q/c, q^2/a^2, q/b, -q/b; q)_n}$$

$$ {}_6\varphi_5 \left[\begin{matrix} a\sqrt{q}, & -a\sqrt{q}, & c, & aq^{n-1/2}/b, & a^2q^{-n-1}, & q^{-n}; & q, & q \\ & q^{n+1}, & a/\sqrt{q}, & -a/\sqrt{q}, & abq^{-n+1/2}, & a^2cq^{-n}/b^2 & & \end{matrix} \right].$$

(5.3.14)

Chapter 6

Some Miscellaneous Transformations of Unilateral and Bilateral Basic Hypergeometric Series

6.1 Introduction

In the present chapter, we derive some miscellaneous transformations for unilateral and bilateral basic hypergeometric series. In particular, in section (6.2) we have obtained some presumably new transformations of basic hypergeometric series using the known summations and transformations available in the literature along with an identity due to Fine [42]. In section (6.3), we have recorded our observation on a recent transformation of Chen and Fu [33] leading to some new transformations of bilateral basic hypergeometric series to unilateral basic hypergeometric series.

The content of this chapter is based on the reference [11].

6.2 Some Transformations of Basic Hypergeometric Series

In his monograph, Fine [42] has recorded a useful identity [eq 20.4; [42]] which can be stated as follows. If

$$g(t) = \sum_{n=0}^{\infty} A_n t^n \tag{6.2.1}$$

then

$$\sum_{n=0}^{\infty} \frac{(aq;q)_n}{(bq;q)_n} A_n t^n = \frac{(aq;q)_\infty}{(bq;q)_\infty} \sum_{k=0}^{\infty} \frac{(b/a;q)_k (aq)^k}{(q;q)_k} g(q^k t). \tag{6.2.2}$$

It may be observed that with a proper choice of A_n in (6.2.1), we may get a transformation and summation from (6.2.2). We have used this fact to established the following transformations which are presumably appear to be new

$$\begin{aligned}
&{}_7\varphi_6\left[\begin{matrix} aq, & \sqrt{a}, & -\sqrt{a}, & \sqrt{b}, & -\sqrt{b}, & \sqrt{bq}, & -\sqrt{bq}; & q, & t \\ & -q, & bq, & \sqrt{c}, & -\sqrt{c}, & \sqrt{cq}, & -\sqrt{cq} & & \end{matrix}\right] \\
&\qquad = \frac{(aq,b;q)_\infty}{(bq,c;q)_\infty} \sum_{k=0}^{\infty} \frac{(b/a;q)_k (aq)^k}{(q;q)_k} \frac{(atq^k;q^2)_\infty}{(tq^k;q^2)_\infty} \\
&\qquad\quad \times {}_3\varphi_2\left[\begin{matrix} c/b, & q^{k/2}\sqrt{t}, & -q^{k/2}\sqrt{t}; & q, & b \\ & q^{k/2}\sqrt{at}, & -q^{k/2}\sqrt{at} & & \end{matrix}\right],
\end{aligned} \tag{6.2.3}$$

where $|t| < 1$ and $|b| < 1$.

$$
{}_3\varphi_2\left[\begin{matrix} a^2, & aq, & aq; & q, & t \\ & bq, & a & & \end{matrix}\right]
$$

$$
= \frac{(aq, a^2tq; q)_\infty}{(bq, t; q)_\infty}\, {}_2\varphi_1\left[\begin{matrix} b/a, & t; & q, & aq \\ & a^2tq & & \end{matrix}\right]
$$

$$
+ \frac{(aq, a^2tq; q)_\infty}{(bq, t; q)_\infty}(at)\, {}_2\varphi_1\left[\begin{matrix} b/a, & t; & q, & aq^2 \\ & a^2tq & & \end{matrix}\right], \quad (6.2.4)
$$

where $|t| < 1$ and $|aq| < 1$.

$$
{}_4\varphi_3\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & aq; & q, & t \\ & \sqrt{a}, & -\sqrt{a}, & bq & & \end{matrix}\right]
$$

$$
= \frac{(aq, atq^2; q)_\infty}{(bq, t; q)_\infty}\, {}_2\varphi_1\left[\begin{matrix} b/a, & t; & q, & aq \\ & atq^2 & & \end{matrix}\right]
$$

$$
- \frac{(aq, atq^2; q)_\infty}{(bq, t; q)_\infty}(at)\, {}_2\varphi_1\left[\begin{matrix} b/a, & t; & q, & aq^2 \\ & atq^2 & & \end{matrix}\right], \quad (6.2.5)
$$

where $|t| < 1$ and $|aq| < 1$.

$$
{}_5\varphi_4\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & aq; & q, & t \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & bq & & \end{matrix}\right]
$$

$$
= \frac{(aq, aq, bt; q)_\infty}{(bq, t, aq/b; q)_\infty} \sum_{k=0}^{\infty} \frac{(b/a, t; q)_k (aq)^k}{(q, bt; q)_k}\, {}_2\varphi_1\left[\begin{matrix} 1/b, & q^k t; & q, & aq \\ & btq^{k+1} & & \end{matrix}\right], \quad (6.2.6)
$$

where $|t| < 1$ and $|aq| < 1$.

$$
{}_4\varphi_3\left[\begin{matrix} aq, & a^2, & ab, & -ab; & q, & t \\ & bq, & a^2b^2, & -ta^2 & & \end{matrix}\right]
$$

$$
= \frac{(aq, at, -at; q)_\infty}{(bq, t, -a^2t; q)_\infty} \sum_{k=0}^{\infty} \frac{(b/a, t, -a^2t; q)_k (aq)^k}{(q, at, -at; q)_k}
$$

$$
{}_2\varphi_2\left[\begin{matrix} a^2, & b^2; & q^2, & a^2t^2q^{2k+1} \\ a^2b^2q, & a^2q^{2k}t^2 & & \end{matrix}\right], \quad (6.2.7)
$$

where $|t| < 1$ and $|a^2t^2q^{2k+1}| < 1$.

$$
{}_3\varphi_2\left[\begin{matrix} aq, & b, & -b; & q, & t \\ & bq, & b^2 & & \end{matrix}\right]
$$

$$
= \frac{(aq, -t; q)_\infty}{(bq; q)_\infty} \sum_{k=0}^{\infty} \frac{(b/a; q)_k (aq)^k}{(q, -t; q)_k}
$$

$$
{}_2\varphi_1\left[\begin{matrix} 0, & 0; & q^2, & t^2q^{2k} \\ & qb^2 & & \end{matrix}\right], \quad (6.2.8)
$$

where $|t| < 1$ and $|t^2q^{2k}| < 1$.

$$
{}_3\varphi_2\left[\begin{matrix} aq, & b, & -b; & q, & t \\ & bq, & b^2 & & \end{matrix}\right]
$$

$$
= \frac{(aq; q)_\infty}{(bq, t; q)_\infty} \sum_{k=0}^{\infty} \frac{(b/a, t; q)_k (aq)^k}{(q; q)_k}
$$

$$
{}_2\varphi_1\left[\begin{matrix} -; & q^2, & b^2t^2q^{2k+1} \\ qb^2 & & \end{matrix}\right], \quad (6.2.9)
$$

where $|t| < 1$ and $|b^2t^2q^{2k+1}| < 1$.

$$
{}_4\varphi_3\left[\begin{matrix} a, & b, & -b, & aq; & q, & t \\ & b^2, & -at, & bq & & \end{matrix}\right]
$$

$$
= \frac{(aq, -t; q)_\infty}{(bq, -at; q)_\infty} \sum_{k=0}^{\infty} \frac{(b/a, -at; q)_k (aq)^k}{(q, -t; q)_k}
$$

$$
{}_2\varphi_1\left[\begin{matrix} a, & aq; & q^2, & t^2q^{2k} \\ & b^2q & & \end{matrix}\right], \quad (6.2.10)
$$

where $|t| < 1$ and $|t^2q^{2k}| < 1$.

$$
{}_3\varphi_2\left[\begin{matrix} a, & b, & aq; & q, & t/ab \\ & t, & bq & & \end{matrix}\right]
$$

$$
= \frac{(aq, t/a, t/b; q)_\infty}{(bq, t, t/ab; q)_\infty} \, {}_3\varphi_2\left[\begin{matrix} b/a, & t, & t/ab; & q, & aq \\ & t/a, & t/b & & \end{matrix}\right], \quad (6.2.11)
$$

where $|t/ab| < 1$ and $|aq| < 1$.

For the proof of (6.2.3) - (6.2.11), we shall require the following known transformations

$$
\sum_{n=0}^{\infty} \frac{(a; q^2)_n (b; q)_{2n}}{(q^2; q^2)_n (c; q)_{2n}} t^n = \frac{(b; q)_\infty (at; q^2)_\infty}{(c; q)_n (t; q^2)_\infty} \sum_{m=0}^{\infty} \frac{(c/b; q)_m (t; q^2)_m}{(q; q)_m (at; q^2)_m} b^m. \quad (6.2.12)
$$

[22]

$$
{}_2\varphi_1\left[\begin{matrix} a^2, & aq; & q, & t \\ & a & & \end{matrix}\right] = (1 + at) \frac{(a^2tq; q)_\infty}{(t; q)_\infty}, \quad (6.2.13)
$$

where $|t| < 1$.

[44]

$$ {}_3\varphi_2\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}; & q, & t \\ & \sqrt{a}, & -\sqrt{a} & & \end{matrix}\right] = \frac{(atq^2;q)_\infty}{(t;q)_\infty}(1-aqt^2), \tag{6.2.14} $$

where $|t| < 1$.

[44]

$$ {}_4\varphi_3\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b; & q, & t \\ & \sqrt{a}, & -\sqrt{a}, & aq/b & & \end{matrix}\right] = \frac{(aq,bt;q)_\infty}{(t,aq/b;q)_\infty} $$

$$ {}_2\varphi_1\left[\begin{matrix} 1/b, & t; & q, & aq \\ & bqt & & \end{matrix}\right], \tag{6.2.15} $$

where $|t| < 1$ and $|aq| < 1$.

[44]

$$ {}_3\varphi_2\left[\begin{matrix} a^2, & ab, & -ab; & q, & t \\ & a^2b^2, & -ta^2 & & \end{matrix}\right] = \frac{(a^2t^2;q^2)_\infty}{(t,-a^2t;q)_\infty} $$

$$ {}_2\varphi_2\left[\begin{matrix} a^2, & b^2; & q^2, & a^2t^2q \\ a^2b^2q, & a^2t^2 & & \end{matrix}\right], \tag{6.2.16} $$

where $|t| < 1$ and $|a^2t^2q| < 1$.

[52]

$$ {}_2\varphi_1\left[\begin{matrix} b, & -b; & q, & t \\ & b^2 & & \end{matrix}\right] = (-t;q)_\infty \, {}_2\varphi_1\left[\begin{matrix} 0, & 0; & q^2, & t^2 \\ & qb^2 & & \end{matrix}\right] \tag{6.2.17} $$

$$= \frac{1}{(t;q)_\infty} {}_0\varphi_1 \left[\begin{matrix} -; & q^2, & qb^2t^2 \\ qb^2 & & \end{matrix} \right]. \qquad (6.2.18)$$

[57]

$${}_3\varphi_2 \left[\begin{matrix} a, & b, & -b; & q, & -t \\ & b^2, & ta & & \end{matrix} \right] = \frac{(t;q)_\infty}{(at;q)_\infty} {}_2\varphi_1 \left[\begin{matrix} a, & aq; & q^2, & t^2 \\ & qb^2 & & \end{matrix} \right]. \qquad (6.2.19)$$

[52]

Replacing t to $-t$ in (6.2.19), we get

$${}_3\varphi_2 \left[\begin{matrix} a, & b, & -b; & q, & t \\ & b^2, & -ta & & \end{matrix} \right] = \frac{(-t;q)_\infty}{(-at;q)_\infty} {}_2\varphi_1 \left[\begin{matrix} a, & aq; & q^2, & t^2 \\ & qb^2 & & \end{matrix} \right], \qquad (6.2.20)$$

where $|t| < 1$.

Proof of (6.2.3). Taking

$$A_n = \frac{(a;q^2)_n(b;q)_{2n}}{(q^2;q^2)_n(c;q)_{2n}},$$

in (6.2.1) and using (6.2.12), we get

$$g(t) = \frac{(b;q)_\infty(at;q^2)_\infty}{(c;q)_\infty(t;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m(t;q^2)_m}{(q;q)_m(at;q^2)_m} b^m.$$

Now by making the use of A_n and $g(t)$ in (6.2.2), we get (6.2.3).

Proof of (6.2.4). Choosing

$$A_n = \frac{(a^2;q)_n(aq;q)_n}{(q;q)_n(a;q)_n},$$

in (6.2.1) and making the use of (6.2.13), we get

$$g(t) = (1 + at)\frac{(a^2tq;q)_\infty}{(t;q)_\infty}.$$

Now by using A_n and $g(t)$ in (6.2.2), we get (6.2.4).

Proof of (6.2.5). Setting

$$A_n = \frac{(a;q)_n(q\sqrt{a};q)_n(-q\sqrt{a};q)_n}{(q;q)_n(\sqrt{a};q)_n(-\sqrt{a};q)_n},$$

in (6.2.1) and using (6.2.14), we get

$$g(t) = (1 - aqt^2)\frac{(atq^2;q)_\infty}{(t;q)_\infty}.$$

Now by substituting A_n and $g(t)$ in (6.2.2), we get (6.2.5).

Proof of (6.2.6). Choosing

$$A_n = \frac{(a;q)_n(q\sqrt{a};q)_n(-q\sqrt{a};q)_n(b;q)_n}{(q;q)_n(\sqrt{a};q)_n(-\sqrt{a};q)_n(aq/b)_n},$$

in (6.2.1) and using (6.2.15), we get

$$g(t) = \frac{(aq,bt;q)_\infty}{(t,aq/b;q)_\infty}\ {}_2\varphi_1\left[\begin{array}{cccc} 1/b, & t; & q, & aq \\ & bqt & & \end{array}\right].$$

Now using A_n and $g(t)$ in (6.2.2), we get (6.2.6).

Proof of (6.2.7). Setting

$$A_n = \frac{(a^2;q)_n(ab;q)_n(-ab;q)_n}{(q;q)_n(a^2b^2;q)_n(-ta^2;q)_n},$$

in (6.2.1) and using (6.2.16), we get

$$g(t) = \frac{(a^2t^2;q^2)_\infty}{(t,-a^2t;q)_\infty}\,{}_2\varphi_2\left[\begin{array}{cccc} a^2, & b^2; & q^2, & a^2t^2q \\ a^2b^2q, & a^2t^2 & & \end{array}\right].$$

Now by making the use of A_n and $g(t)$ in (6.2.2), we get (6.2.7).

Proof of (6.2.8). Taking

$$A_n = \frac{(b;q)_n(-b;q)_n}{(q;q)_n(b^2;q)_n},$$

in (6.2.1) and using (6.2.17), we get

$$g(t) = (-t;q)_\infty\,{}_2\varphi_1\left[\begin{array}{cccc} 0, & 0; & q^2, & t^2 \\ & qb^2 & & \end{array}\right].$$

Now by substituting A_n and $g(t)$ in (6.2.2), we get (6.2.8).

Proof of (6.2.9). Choosing same

$$A_n = \frac{(b;q)_n(-b;q)_n}{(q;q)_n(b^2;q)_n},$$

in (6.2.1) and using (6.2.18), we get

$$g(t) = \frac{1}{(t;q)_\infty} \, {}_0\varphi_1 \left[\begin{matrix} -; & q^2, & qb^2t^2 \\ qb^2 & & \end{matrix} \right].$$

Now by making the use of A_n and $g(t)$ in (6.2.2), we get (6.2.9).

Proof of (6.2.10). Taking

$$A_n = \frac{(a;q)_n(b;q)_n(-b;q)_n}{(q;q)_n(b^2;q)_n(-at;q)_n},$$

in (6.2.1)and using (6.2.20), we get

$$g(t) = \frac{(-t;q)_\infty}{(-at;q)_\infty} \, {}_2\varphi_1 \left[\begin{matrix} a, & aq; & q^2, & t^2 \\ & qb^2 & & \end{matrix} \right].$$

Now by using A_n and $g(t)$ in (6.2.2), we get (6.2.10).

Proof of (6.2.11). Setting

$$A_n = \frac{(a;q)_n(b;q)_n}{(q;q)_n(t;q)_n} \left(\frac{1}{ab} \right)^n,$$

in (6.2.1) and using (1.3.11), we get

$$g(t) = \frac{(t/a;q)_\infty(t/b;q)_\infty}{(t;q)_\infty(t/ab;q)_\infty}.$$

Now by putting A_n and $g(t)$ in (6.2.2), we get (6.2.11).

6.3 Some Transformations from Bilateral Basic hypergeometric to Unilateral Basic Hypergeometric Series

In this section we have recorded some transformations of bilateral basic hypergeometric series into unilateral basic hypergeometric geometric series which are direct consequences of the following identity due to Chen et al [33].

$$
\begin{aligned}
&{}_2\psi_2\left[\begin{matrix} a, & b; & q, & z \\ c, & d & & \end{matrix}\right] \\
&= \frac{(c/b, abz/d, dq/abz, q/d, q; q)_\infty}{(c, az/d, q/a, q/b, cd/abz; q)_\infty}\, {}_2\varphi_1\left[\begin{matrix} d/a, & cd/abz; & q, & bq/d \\ & dq/az & & \end{matrix}\right] \\
&- \frac{(az, q/az, cq/d, b, d/a, q; q)_\infty}{(c, d, bq/d, az/d, dq/az, q/a; q)_\infty}\frac{(az)}{(d)}\, {}_2\varphi_1\left[\begin{matrix} aq/d, & bq/d; & q, & z \\ & cq/d & & \end{matrix}\right] \qquad (6.3.1)
\end{aligned}
$$

In the next section we shall need the following known transformations [44].

$$
{}_2\varphi_1\left[\begin{matrix} a, & b; & q, & z \\ & c & & \end{matrix}\right] = \frac{(b, az; q)_\infty}{(c, z; q)_\infty}\, {}_2\varphi_1\left[\begin{matrix} c/b, & z; & q, & b \\ & az & & \end{matrix}\right]. \qquad (6.3.2)
$$

$$
{}_2\varphi_1\left[\begin{matrix} a, & b; & q, & z \\ & c & & \end{matrix}\right] = \frac{(c/b, bz; q)_\infty}{(c, z; q)_\infty}\, {}_2\varphi_1\left[\begin{matrix} abz/c, & b; & q, & c/b \\ & bz & & \end{matrix}\right]. \qquad (6.3.3)
$$

$$
{}_2\varphi_1\left[\begin{matrix} a, & b; & q, & z \\ & c & & \end{matrix}\right] = \frac{(abz/c; q)_\infty}{(z; q)_\infty}\, {}_2\varphi_1\left[\begin{matrix} c/a, & c/b; & q, & abz/c \\ & c & & \end{matrix}\right]. \qquad (6.3.4)
$$

$$
{}_2\varphi_1\left[\begin{matrix} a, & b; & q, & z \\ & c & & \end{matrix}\right] = \frac{(az;q)_\infty}{(z;q)_\infty}\,{}_2\varphi_2\left[\begin{matrix} a, & c/b; & q, & bz \\ c, & az & & \end{matrix}\right]. \qquad (6.3.5)
$$

$$
{}_2\varphi_1\left[\begin{matrix} a, & b; & q, & z \\ & c & & \end{matrix}\right] = \frac{(abz/c;q)_\infty}{(bz/c;q)_\infty}\,{}_3\varphi_2\left[\begin{matrix} a, & c/b, & 0; & q, & q \\ & c, & cq/bz & & \end{matrix}\right]
$$

$$
+\frac{(a,bz,c/b;q)_\infty}{(c,z,c/bz;q)_\infty}\,{}_3\varphi_2\left[\begin{matrix} z, & abz/c, & 0; & q, & q \\ & bz, & bzq/c & & \end{matrix}\right]. \qquad (6.3.6)
$$

And the Ramanujan ${}_1\psi_1$ sum [66]

$$
{}_1\psi_1\left[\begin{matrix} a; & q, & z \\ b & & \end{matrix}\right] = \frac{(az,q/az,b/a,q;q)_\infty}{(z,b/az,q/a,b;q)_\infty}, \qquad (6.3.7)
$$

where $|b/a| < |z| < 1$.

Finally, we have derived the following identities which are true whenever the series involved are convergent

$$
{}_2\psi_2\left[\begin{matrix} a, & b; & q, & z \\ c, & abz & & \end{matrix}\right]
$$

$$
= \frac{(az,bz,q/bz,q;q)_\infty}{(c,z,q/b,abz;q)_\infty}\quad {}_2\varphi_1\left[\begin{matrix} c/a, & z; & q, & q/bz \\ & q/a & & \end{matrix}\right], \qquad (6.3.8)
$$

where $|cz| < |z| < 1$ and $|q/bz| < 1$.

$$
{}_2\psi_2\left[\begin{matrix} a, & b; & q, & z \\ c, & abz & & \end{matrix}\right]
$$

$$= \frac{(az, bz, q, c/a; q)_\infty}{(c, q/a, abz, z; q)_\infty} \quad {}_2\varphi_1 \left[\begin{matrix} q/c, & q/bz; & q, & c/a \\ & q/b & & \end{matrix} \right], \quad (6.3.9)$$

where $|cz| < |z| < 1$ and $|c/a| < 1$.

$${}_2\psi_2 \left[\begin{matrix} a, & b; & q, & ,z \\ c, & abz & & \end{matrix} \right]$$

$$= \frac{(az, q/c, bz, q, cq/abz; q)_\infty}{(c, abz, q/a, q/b, z; q)_\infty} \quad {}_2\varphi_1 \left[\begin{matrix} c/a, & c/b; & q, & q/c \\ & cq/abz & & \end{matrix} \right], \quad (6.3.10)$$

where $|cz| < |z| < 1$ and $|q/c| < 1$.

$${}_2\psi_2 \left[\begin{matrix} a, & b; & q, & z \\ c, & abz & & \end{matrix} \right]$$

$$= \frac{(az, cq/abz, bz, q; q)_\infty}{(c, q/b, abz, z; q)_\infty} \quad {}_2\varphi_2 \left[\begin{matrix} q/az, & c/a; & q, & q/b \\ q/a, & cq/abz & & \end{matrix} \right], \quad (6.3.11)$$

where $|cz| < |z| < 1$ and $|q/b| < 1$.

$${}_2\psi_2 \left[\begin{matrix} a, & b; & q, & z \\ c, & abz & & \end{matrix} \right]$$

$$= \frac{(az, cq/abz, bz, q/c, q; q)_\infty}{(c, q/a, q/b, az/c, abz; q)_\infty} \quad {}_3\varphi_2 \left[\begin{matrix} q/az, & c/a, & 0; & q, & q \\ & cq/az, & cq/abz & & \end{matrix} \right]$$

$$+ \frac{(az, bz, q, q/az, c/a; q)_\infty}{(q/a, c, abz, z, c/az; q)_\infty} \quad {}_3\varphi_2 \left[\begin{matrix} z, & q/c, & 0; & q, & q \\ & q/b, & azq/c & & \end{matrix} \right], \quad (6.3.12)$$

where $|cz| < |z| < 1$.

$$\sum_{n=0}^{\infty} \frac{(b/a;q)_n q^n}{(bq^2/c;q)_n} = \frac{(c/bq;q)_\infty (c/a;q)_\infty}{(c/b;q)_\infty (c/aq;q)_\infty} + \frac{c}{bq} \frac{(b/a;q)_\infty (c/a;q)_\infty}{(bq^2/c;q)_\infty (c/aq;q)_\infty} \tag{6.3.13}$$

Proof of (6.3.8). Putting $d = abz$ in (6.3.1) we get,

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n (b;q)_n z^n}{(c;q)_n (abz;q)_n}$$

$$= \frac{(az, cq/abz, bz, q;q)_\infty}{(q/a, q/b, c, abz;q)_\infty} \sum_{n=0}^{\infty} \frac{(q/bz;q)_n (q/az;q)_n z^n}{(q;q)_n (cq/abz;q)_n}. \tag{6.3.14}$$

Changing $a = q/az$, $b = q/bz$ and $c = cq/abz$ in (6.3.2), we deduce

$$\sum_{n=0}^{\infty} \frac{(q/az;q)_n (q/bz;q)_n z^n}{(q;q)_n (cq/abz;q)_n}$$

$$= \frac{(q/bz, q/a;q)_\infty}{(cq/abz, z;q)_\infty} \sum_{n=0}^{\infty} \frac{(c/a;q)_n (z;q)_n}{(q;q)_n (q/a;q)_n} \left(\frac{q}{bz}\right)^n. \tag{6.3.15}$$

Substituting (6.3.15) in (6.3.14), we have (6.3.8).

Proof of (6.3.9). Taking $a = q/az$, $b = q/bz$ and $c = cq/abz$ in (6.3.3), we obtain

$$\sum_{n=0}^{\infty} \frac{(q/az;q)_n (q/bz;q)_n z^n}{(q;q)_n (cq/abz;q)_n}$$

$$= \frac{(c/a, q/b;q)_\infty}{(cq/abz, z;q)_\infty} \sum_{n=0}^{\infty} \frac{(q/c;q)_n (q/bz;q)_n}{(q;q)_n (q/b;q)_n} \left(\frac{c}{a}\right)^n. \tag{6.3.16}$$

Using (6.3.16) in (6.3.14), we obtain (6.3.9).

Proof of (6.3.10). Setting $a = q/az$, $b = q/bz$ and $c = cq/abz$ in (6.3.4), we have

$$\sum_{n=0}^{\infty}\frac{(q/az;q)_n(q/bz;q)_n z^n}{(q;q)_n(cq/abz;q)_n} = \frac{(q/c;q)_\infty}{(z;q)_\infty}\sum_{n=0}^{\infty}\frac{(c/a;q)_n(c/b;q)_n}{(q;q)_n(cq/abz;q)_n}\left(\frac{q}{c}\right)^n. \tag{6.3.17}$$

On employing (6.3.17)in (6.3.14), we obtain (6.3.10).

Proof of (6.3.11). Substituting $a = q/az$, $b = q/bz$ and $c = cq/abz$ in (6.3.5), we get

$$\sum_{n=0}^{\infty}\frac{(q/az;q)_n(q/bz;q)_n z^n}{(q;q)_n(cq/abz;q)_n}$$

$$= \frac{(q/a;q)_\infty}{(z;q)_\infty}\ {}_2\varphi_2\left[\begin{matrix} q/az, & c/a; & q, & q/b \\ q/a, & cq/abz & & \end{matrix}\right] \tag{6.3.18}$$

by using (6.3.18) in (6.3.14), we have (6.3.11).

Proof of (6.3.12). Substituting $a = q/az$, $b = q/bz$ and $c = cq/abz$ in (6.3.6), we get

$$\sum_{n=0}^{\infty}\frac{(q/az;q)_n(q/bz;q)_n z^n}{(q;q)_n(cq/abz;q)_n}$$

$$= \frac{(q/c;q)_\infty}{(az/c;q)_\infty}\sum_{n=0}^{\infty}\frac{(q/az;q)_n(c/a;q)_n(0;q)_n(q)^n}{(q;q)_n(cq/abz;q)_n(cq/az;q)_n}$$

$$+\frac{(q/az,q/b,c/a;q)_\infty}{(cq/abz,z,c/az;q)_\infty}\sum_{n=0}^{\infty}\frac{(z;q)_n(q/c;q)_n(0;q)_n(q)^n}{(q;q)_n(q/b;q)_n(azq/c;q)_n}. \tag{6.3.19}$$

by using (6.3.19) in (6.3.14), we have (6.3.12).

Proof of (6.3.13). Substituting $d = b$ in (6.3.1), we get

$$
{}_1\psi_1\left[\begin{matrix} a; & q, & z \\ c & & \end{matrix}\right]
$$

$$
= \frac{(c/b, az, q/az, q; q)_\infty}{(c, az/b, q/a, c/az; q)_\infty} \quad {}_2\varphi_1\left[\begin{matrix} b/a & c/az & ;q & ;q \\ bq/az & & & \end{matrix}\right]
$$

$$
- \frac{(az, q/az, cq/b, b/a; q)_\infty}{(c, az/b, bq/az, q/a; q)_\infty}\frac{(az)}{(b)} \quad {}_2\varphi_1\left[\begin{matrix} aq/b & q & ;q & ;z \\ cq/b & & & \end{matrix}\right] \tag{6.3.20}
$$

now using (6.3.7) on left side of the (6.3.20) and then replace $z = c/aq$ and finally using (1.3.11), we get (6.3.13). If we take $c = bq$ in (6.3.13), we get well-known q-binomial theorem.

The identities (6.3.8) - (6.3.13) contain a number of known results as their special cases. They can also be used to derive eta function identities. For example, if we set $a = q^{1/2}, b = q^{3/2}, c = q^3$ and then replace $q = q^2$ in (6.3.13) and using (6.3.22), we get

$$
\sum_{n=0}^{\infty}\frac{(q^2;q^2)_n q^{2n}}{(q;q^2)_n} = \frac{(1-q)}{(1-q^3)} + \frac{q^{7/8}}{(1-q^3)}\frac{\eta^2(2\tau)}{\eta(\tau)}. \tag{6.3.21}
$$

$\eta(\tau)$ is Dedekind eta function defined as

$$
\eta(\tau) = e^{\pi i\tau/12}\prod_{n=0}^{\infty}(1 - e^{2\pi i\kappa\tau}) = q^{1/24}(q;q)_\infty, \tag{6.3.22}
$$

where $q = e^{2\pi i\tau}$ and $Im(\tau) > 0$.

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